On groups with linear sci growth

by

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Abstract. We prove that the semistability growth of hyperbolic groups is linear, which implies that hyperbolic groups which are sci (simply connected at infinity) have linear sci growth. Based on the linearity of the end-depth of finitely presented groups we show that the linear sci is preserved under amalgamated products over finitely generated one-ended groups. Eventually one proves that most non-uniform lattices have linear sci.

1. Introduction. The metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there are constants λ, C and maps $f : X \to Y$, $g : Y \to X$ (called (λ, C)-*quasi-isometries*) such that

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C, \quad d_X(gf(x), x) \leq C,$$
$$d_X(g(y_1), g(y_2)) \leq \lambda d_Y(y_1, y_2) + C, \quad d_Y(fg(y), y) \leq C,$$

for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$.

DEFINITION 1.1. A connected locally compact locally simply connected topological space X with $\pi_1 X = 0$ is *simply connected at infinity* (abbreviated *sci*, and one also writes $\pi_1^\infty X = 0$) if for each compact $k \subseteq X$ there exists a larger compact $k \subseteq K \subseteq X$ such that any closed loop in $X \smallsetminus K$ is null homotopic in $X \smallsetminus k$.

The sci is a fundamental tameness condition for non-compact spaces, as it singles out Euclidean spaces among contractible manifolds, following classical results of Stallings and Siebenmann. The concept of sci of finitely presented groups goes back at least to Siebenmann's thesis [25] and seems to first appear in its present form in Houghton's paper [13] (see also [19, 4]), as follows:

DEFINITION 1.2. A finitely presented group G is *simply connected at infinity* (abbreviated *sci*) if for some (equivalently any) finite complex X_G with $\pi_1 X_G = G$, its universal covering $\widetilde{X_G}$ is sci.

The group $\mathbb{Z}^2$ is obviously not sci. More interestingly, M. Davis (see e.g. [9]) constructed word hyperbolic groups G (of virtual cohomological dimension $n \geq 4$ by the results of Bestvina and Mess from [1]) which are not sci.

All groups considered here will be finitely presented (unless otherwise explicitly stated), and a system of generators determines a word metric on the group. Although this depends on the chosen generating set, different word metrics are quasi-isometric. In [10] we enhanced the topological sci notion in the case of groups by taking advantage of this metric structure.

DEFINITION 1.3. Let X be a sci non-compact metric space. The *sci growth* $V_X(r)$ (called rate of vanishing of π_1^∞ in [10]) is the infimal $N(r)$ with the property that any loop in the complement of the metric ball $B(N(r))$ of radius $N(r)$ (centered at the identity) bounds a 2-disk outside $B(r)$.

REMARK 1.4. It is easy to construct examples of metric spaces with arbitrarily large V_X.

It is customary to introduce the following (rough) equivalence relation on real valued functions: the real functions f and g are equivalent, denoted by $f \sim g$, if there exist constants c_i, C_j for $i = 1, 2, 3$ (with $c_1, c_2 > 0$) such that:
$$c_1 f(c_2 x) + c_3 \leq g(x) \leq C_1 f(C_2 x) + C_3 \quad \text{for all } x.$$
It is proved in [10] that the (rough) equivalence class of $V_X(r)$ is a quasi-isometry invariant. In particular, if a finitely presented group G is sci, then the (rough) equivalence class of the real function $V_G = V_{\widetilde{X_G}}$ is a quasi-isometry invariant of G, where $\widetilde{X_G}$ is the universal covering space of any finite complex X_G with $\pi_1(X_G) = G$.

If a finitely presented group G is sci and V_G is a linear function, we will say that G has *linear sci*. In contrast with the abundance of equivalence classes of geometric invariants of finitely presented groups (like group growth, Dehn functions or isodiametric functions), the metric refinements of topological properties seem highly constrained. We already found in [10] that many cocompact lattices in Lie groups and in particular geometric 3-manifold groups have linear sci. The aim of this paper is to further explore this phenomenon by considerably enlarging the class of groups with linear sci. Our first result is:

THEOREM 1.5. *Word hyperbolic groups which are sci have linear sci.*

Let us recall that a group G is *one-ended* (or 0-*connected at infinity*) if, for any compact subset L of the Cayley graph X_G of G, there exists a compact subset $K \supset L$ such that any two points outside K can be joined by a path contained in $X_G \setminus L$.

If H is a subgroup of two groups G_1 and G_2, the *amalgamated product* $G_1 *_H G_2$ is the quotient of the free product of G_1 and G_2 where the copies of H in G_1 and G_2 are identified. If H and K are isomorphic subgroups of G_1, the *HNN-extension* $G_1 *_H$ is the quotient of the free product $G_1 * \langle t \rangle$, where H is identified with $t^{-1} K t$ and $\langle t \rangle$ denotes the free cyclic group generated by t.

We next show that the class of groups with linear sci is closed under amalgamated free products known to preserve the sci.

THEOREM 1.6.

(1) *Let G_1 and G_2 be one-ended finitely presented groups with linear sci, and H be a finitely generated subgroup of G_1 and G_2 with one end. Then the amalgamated free product $G = G_1 *_H G_2$ has linear sci.*

(2) *Let G_1 be a finitely presented group with linear sci, and H, K be isomorphic finitely generated subgroups with one end. Then the HNN-extension $G = G_1 *_H$ has linear sci.*

Theorem 1.6 is similar to (but subject to stronger restrictions than) the results obtained by Mihalik and Tschantz [21, 22] in the context of semistability. Notice that the sci is *not* preserved under amalgamated products over multi-ended subgroups (see [14]).

Previous results of [10] dealt with all cocompact lattices in connected Lie groups but the solvable ones. We now consider non-uniform lattices. Our main result in this direction is:

THEOREM 1.7.

(1) *Let G be a semisimple Lie group for which the associated symmetric space G/K is of dimension $n \geq 4$ and of $\mathbb{R}$-rank greater than or equal to 2. Let Γ be an irreducible, non-uniform lattice in G of $\mathbb{Q}$-rank one. Then Γ is sci with linear sci.*

(2) *Every lattice $\Gamma \subset \mathrm{SO}(n, 1)$, $n \geq 2$, has linear sci.*

REMARK 1.8. We believe that the last result also holds for $\mathbb{R}$-rank 1 semisimple Lie groups and for non-uniform lattices of $\mathbb{Q}$-rank > 1. Further, if moreover strongly polycyclic groups had linear sci, then all lattices in Lie groups (of sufficiently large dimension) would have linear sci.

The results of this paper naturally lead to the question of the existence of sci groups with superlinear sci growth. It seems still unknown whether sci CAT(0) groups have linear sci.

2. Proof of Theorem 1.5

2.1. Preliminaries on hyperbolic groups. Let (X, d) be a geodesic metric space, which in our case will be the Cayley graph of a finitely generated group G endowed with the word metric induced by a finite generating system. Let γ be a geodesic path in X, possibly infinite. For any $x, y \in \gamma$, we denote by $[x, y]_\gamma$ the subpath of γ that connects x with y. When γ is finite, we denote by $\ell(\gamma)$ the length of the path γ.

A geodesic triangle in X is δ-*slim* if every side is contained in the δ-neighborhood of the union of the other sides. The group G is δ-*hyperbolic*, for some $\delta \geq 0$, if all geodesic triangles in X are δ-slim. The group G is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. It is well known that hyperbolic groups are finitely presented.

Suppose from now on that G is a δ-hyperbolic group and X_G is its Cayley complex associated with a finite presentation $\mathcal{P} = \langle S \mid R \rangle$. We will only consider geodesics within the Cayley graph, that is, the 1-skeleton $X_G^{(1)}$ of X_G. Notice that while the Cayley complex may change when we add words equal to the identity to the relators in $\mathcal{P}$, the Cayley graph remains unchanged.

Bestvina and Mess [1] proved the following crucial fact, which was pointed out by Mihalik:

PROPOSITION 2.1 ([1]). *Let G be a hyperbolic one-ended group. There is a constant $c \geq 0$ such that for all $x \in X_G$ there exists an infinite geodesic ray issuing from the identity of G which passes within c of x.*

We say that two geodesic rays are *asymptotic* if their images in X_G are a finite Hausdorff distance apart. This defines an equivalence relation on the collection of geodesic rays in X_G. The *boundary* ∂X_G of X_G is the collection of equivalence classes, under this relation, of geodesic rays in X_G. Unless otherwise stated, all geodesics considered will be assumed to be unit speed geodesics.

We say that X_G *satisfies* $\mathrm{Rel}(M)$ for some $M > 0$ if there exists $L > 0$ such that for all $R > 0$ and $x, y \in X_G$ with $d(x, 1) = d(y, 1) = R$ and $d(x, y) \leq M$ there exists a path of length at most L that connects x and y outside the ball $B(R - c)$, where c is the constant of Proposition 2.1. It will be convenient to say that then X_G satisfies $\mathrm{Rel}(M)$ *with constant L*. The significance of $\mathrm{Rel}(M)$ is revealed in the following:

PROPOSITION 2.2 ([1]). *Let G be a hyperbolic one-ended group. If $\mathrm{Rel}(M)$ fails for some $M > 0$, then ∂G contains a global cut point.*

Combining this with a result of Bowditch, Svenson and Swarup [3, 27, 26] which states that ∂X_G has no global cut points, we derive that every one-ended hyperbolic group satisfies $\mathrm{Rel}(M)$ for any $M > 0$.

2.2. Semistability and simple connectivity at infinity. The aim of this section is to put the sci growth into a more general context. Recall that a *ray* in a non-compact topological space X is a proper map $\gamma : [0, \infty) \to X$. As above we consider X to be a connected locally compact locally simply connected topological space. Two rays γ_1 and γ_2 converge to the *same end* of X if for any compact $C \subset X$ there exists R such that $\gamma_1([R, \infty))$ and $\gamma_2([R, \infty))$ lie in the same component of $X \smallsetminus C$. The set of rays under this equivalence relation is the same as the set of ends of X.

DEFINITION 2.3. An end of X is *semistable* if any two rays of X converging to this end are properly homotopic. This is equivalent (see e.g. [11]) to the following: for any ray γ converging to the end and for any $n \geq 0$, there exists $N \geq n$ such that any loop based on a point of γ with image outside the metric ball $B(N)$ of radius N and fixed center can be pushed (rel γ) to infinity by a homotopy in $X \smallsetminus B(n)$.

A topological space is *semistable* if all its ends are semistable. This definition was extended to groups: A finitely presented group G is *semistable* if for some (equivalently any) finite complex X_G with $\pi_1(X_G) = G$ its universal covering $\widetilde{X_G}$ is semistable.

Many classes of groups are known to be semistable (see e.g. [21, 20] and also [3, 27, 26] for the case of hyperbolic groups) but examples of finitely presented groups which are *not* semistable are still unknown. There is a well-defined notion of topological fundamental group at infinity associated to a semistable end of a group (see [12]). Now, much as in [10], we consider the following metric refinement of semistability:

DEFINITION 2.4. Let X be a non-compact metric space, e an end of X, and γ a ray converging to e. The *semistability growth* function $S_e(r)$ is the infimal $N(r)$ with the following property: for any $R \geq N$ and any loop l based on γ which lies in $X \smallsetminus B(N)$ there exists a homotopy rel γ supported in $X \smallsetminus B(r)$ which moves l to a loop in $X \smallsetminus B(R)$.

Set S_G for $\sup_e S_e$, with e running over the set of ends of $\widetilde{X_G}$, where X_G is a finite complex with fundamental group G, whenever this is defined. It is not difficult to see that the equivalence class of S_G is a well-defined quasi-isometry invariant of the finitely presented group G.

The principal result of this section is the following immediate connection between sci growth and semistability growth:

PROPOSITION 2.5. *Assume that G is a finitely presented sci group. Then* $V_G = S_G$.

Proof. For given r, as the space $\widetilde{X_G}$ is sci there exists some large enough $N(r)$ so that any loop within $\widetilde{X_G} \smallsetminus B(N(r))$ bounds a disk outside $B(r)$.

Let l be a loop not intersecting $B(S_G(r))$. By the semistability assumption one can homotope l in $\widetilde{X_G} \smallsetminus B(r)$ to a loop l' lying within $\widetilde{X_G} \smallsetminus B(N(r))$. But l' bounds a disk outside $B(r)$, and hence l bounds a disk outside $B(r)$. This proves that $V_G(r) \leq S_G(r)$.

For the reverse inequality let l be a loop based at $\gamma(V_G(r) + \varepsilon)$ (for arbitrarily small ε) outside $B(V_G(r))$, where γ is a given ray. Then l bounds a disk outside $B(r)$, which yields a nullhomotopy of the based loop l to the base point p. We then push p along γ as far as we want. This proves that $S_G(r) \leq V_G(r)$. ∎

It follows that Theorem 1.5 is an immediate consequence of the more general

THEOREM 2.6. *The semistability growth of word hyperbolic groups is linear.*

2.3. Proof of Theorem 2.6. Consider first the case when G is a one-ended hyperbolic group. Let δ be the hyperbolicity constant for a Cayley complex X_G of G, and let c be the constant provided by Proposition 2.1. By Proposition 2.2 and the absence of cut points (see [3, 27, 26]), X_G satisfies $\mathrm{Rel}(M)$ for some $M > 6c + 2\delta + 3$ with constant $L > 2c + 4$. Without loss of generality we can assume that X_G is associated with a presentation of G that contains as relators all words of length less than $2L + 4c$ which are equal to the identity in G. In this section, unless stated otherwise, the balls we consider will be centered at the identity.

Let $n \in \mathbb{Z}_+$ and γ be a geodesic ray in X_G that starts from the identity. We will show that every loop f based at a point $x \in \gamma$ and which lies outside $B(n + 2c)$ can be pushed rel γ arbitrarily far away by a homotopy outside $B(n)$.

Let p, q be two adjacent vertices of f, and $r = d(p, 1) > n + 2c$. There exist unit speed geodesic rays γ_0 and γ_1 issuing from the identity which pass within c of p and q, respectively.

Following [1, Prop. 3.2] we now establish:

LEMMA 2.7. *For every integer $i \geq 0$, there exists a path f_i from $\gamma_0(r+i)$ to $\gamma_1(r+i)$ such that:*

(1) *The path f_i lies outside $B(r + i - c)$.*
(2) *For any $j \in \{0, 1, \ldots, L^i\}$, there is a unit speed geodesic ray γ_{j/L^i} issuing from the identity such that*
$$\gamma_{j/L^i}(r+i) \in f_i,$$
and for $j < L^i - 1$,
$$d(\gamma_{j/L^i}(r+i), \gamma_{(j+1)/L^i}(r+i)) \leq M.$$

Proof. We use induction on i. From the triangle inequality we obtain $d(\gamma_0(r), p) \leq 2c$ and $d(\gamma_1(r), q) \leq 2c+1$, so that $d(\gamma_0(r), \gamma_1(r)) \leq 4c+2 < M$. By property Rel($M$), there is a path f_0 of length at most L which joins $\gamma_0(r)$ to $\gamma_1(r)$ outside $B(r-c)$, hence the claim holds for $i = 0$.

Assume now that the result holds for some $i \geq 0$. By property Rel(M), for any $j \in \{0, 1, \ldots, L^i - 1\}$, there is a path $\alpha_j : [0, L] \to X_G$, of at most unit speed, that joins $\gamma_{j/L^i}(r+i)$ to $\gamma_{(j+1)/L^i}(r+i)$ and lies outside $B(r+i-c)$.

Next, for any $k \in \{1, \ldots, L-1\}$, there exists a geodesic ray γ_k' issuing from 1 that passes within c of $\alpha_j(k)$. Let $y_k \in \gamma_k'$ be a closest point to $\alpha_j(k)$. Then $d(y_k, \alpha_j(k)) \leq c$, so that $d(y_k, 1) \geq r + i - 2c$. It follows that there exists $z_k \in \gamma_k'$ with $d(z_k, 1) \geq r + i + 1$ and $d(z_k, y_k) \leq 2c + 1$. We therefore obtain $d(z_k, z_{k+1}) \leq 6c + 3$. As the geodesic triangle of vertices $1, z_k, z_{k+1}$ is δ-slim, we derive that

$$d(\gamma_k'(r+i+1), \gamma_{k+1}'(r+i+1)) \leq 6c + 2\delta + 3 < M.$$

Therefore, from Rel(M) there exists a path of length at most L which lies outside $B(r+i+1-c)$ and joins $\gamma_k'(r+i+1)$ to $\gamma_{k+1}'(r+i+1)$. The concatenation of these paths yields the desired f_{i+1}. We set $\gamma_{(jL+k)/L^{i+1}}$ to be the geodesic ray γ_k', thereby concluding the induction step. Notice that

$$\ell([\gamma_{j/L^i}(r+i), \gamma_{(j+1)/L^i}(r+i)]_{f_i}) \leq L. \blacksquare$$

Let P, Q be two geodesic arcs joining p and q to $\gamma_0(r)$ and $\gamma_1(r)$, respectively. For any $N \geq 0$, let $\Phi_N(p, q)$ be the closed oriented loop obtained by concatenation of P, $\gamma_0([r, r+N])$, f_N, $\gamma_1([r, r+N])^{-1}$, Q^{-1} and the edge qp.

LEMMA 2.8. *The loop $\Phi_N(p, q)$ is null homotopic outside $B(n)$ for $N \geq 0$.*

Proof. The closed loop $\Phi_0(p, q)$ has length at most $L + 4c + 2$ and lies outside $B(r-c-1)$. By our hypothesis on the group presentation, this loop bounds a 2-cell A_0 in the Cayley complex, thereby proving the claim for $N = 0$.

Now, for any $i \in \{0, 1, \ldots, N-1\}$, let Φ_i be the concatenation of the four paths $\gamma_0([r+i, r+i+1])$, f_{i+1}, $\gamma_1([r+i, r+i+1])^{-1}$, and f_i^{-1}. Then $\Phi_N(p, q)$ is homotopic to the product of $\Phi_0(p, q)$ and $\Phi_0 \Phi_1 \cdots \Phi_{N-1}$. We can further decompose each Φ_i as the composition of loops $\Phi_i(j)$ consisting of the concatenation of the following four paths: $\gamma_{j/L^i}([r+i, r+i+1])$, $[\gamma_{j/L^i}(r+i+1), \gamma_{(j+1)/L^i}(r+i+1)]_{f_{i+1}}$, $\gamma_{(j+1)/L^i}([r+i, r+i+1])^{-1}$, and $[\gamma_{j/L^i}(r+i), \gamma_{(j+1)/L^i}(r+i)]_{f_i}^{-1}$.

Recall from above that $\ell([\gamma_{j/L^i}(r+i), \gamma_{(j+1)/L^i}(r+i)]_{f_i}) \leq L$. For fixed i and j, let $\alpha_j : [0, L] \to X_G$ denote a path with image

$$a_j([0, L]) = [\gamma_{j/L^i}(r+i), \gamma_{(j+1)/L^i}(r+i)]_{f_i}.$$

Then a_j has at most unit speed and lies outside $B(r+i-c)$. For any k in $\{0, 1, \ldots, L-1\}$, denote by y_k a point on the geodesic $\gamma_k' = \gamma_{(jL+k)/L^{i+1}}$

which is closest to $a_j(k)$, and by β_k a geodesic path that joins $\alpha_j(k)$ to y_k. Then $d(y_k, a_j(k)) \leq c$ and

$$r + i - 2c \leq d(y_k, 1) \leq L/2 + r + i + c.$$

Hence $d(y_k, \gamma_k'(r+i+1)) \leq L/2+c+1$ and the path $[y_k, \gamma_k'(r+i+1)]_{\gamma_k'}$ lies outside $B(r+i-2c)$. Therefore, the closed loop obtained by concatenating β_k, $[y_k, \gamma_k'(r+i+1)]_{\gamma_k'}$, $[\gamma_k'(r+i+1), \gamma_{k+1}'(r+i+1)]_{f_{i+1}}$, $[\gamma_{k+1}'(r+i+1), y_{k+1}]_{\gamma_{k+1}'}$, β_{k+1}, and $[\alpha_j(k), \alpha_j(k+1)]_{\alpha_j}$ has length at most $2L+4c-1$ and lies outside $B(r+i-2c)$. So, it bounds a 2-cell $A_{i,j}(k)$ in the Cayley complex outside $B(n)$. The union $A_{i,j} = \bigcup_{k \in \{0,1,\ldots,L-1\}} A_{i,j}(k)$ of all these 2-cells is the image of a disk filling the loop $\Phi_i(j)$ outside $B(n)$. This proves the lemma. ∎

The composition of all loops of the $\Phi_N(p,q)$ for p, q successive vertices of the loop f is then freely homotopic within $X_G \setminus B(n)$ to a loop consisting of concatenations of paths of the form f_N which lie outside $B(r+N-c)$. It follows that for any $N \geq 0$ our initial loop $f \subset X_G \setminus B(n+2c)$ is homotopic rel γ to a loop in $X_G \setminus B(r+N-c)$ by a homotopy outside $B(n)$.

Eventually, when G is not one-ended, we work on the connected components of $X_G \setminus B(n+2c)$. This proves Theorem 2.6. Another proof can be given following the arguments from [2, 15, 16].

3. Proof of Theorem 1.6

3.1. Preliminaries on the end-depth. The sci and its refinement (the sci growth rough equivalence class) are 1-dimensional invariants at infinity for a group G, in the sense that they take care of loops and disks. The 0-dimensional analogue of the simple connectivity at infinity is the connectivity at infinity, namely the one-endedness. One could adapt the notion of sci growth to the growth of an end—already considered by Cleary and Riley (see [6]). This leads to the following metric refinement which is the 0-dimensional counterpart of the sci growth:

DEFINITION 3.1. Let X be a one-ended metric space. The *end-depth* $V_0(X)$ of X is the infimal $N(r)$ with the property that any two points which sit outside the ball $B(N(r))$ of radius $N(r)$ can be joined by a path outside $B(r)$.

If G is a finitely generated one-ended group, then the *end-depth* of G is the (rough) equivalence class of the real function $V_{0,G} = V_0(X_G)$, where X_G is the Cayley graph of G associated to a finite generating set.

In the same way one can define the end-depth of a specific end of a space or finitely generated group which are not necessarily one-ended. In [23] it is proved that the (rough) equivalence class of $V_{0,G}$ is a well-defined quasi-isometry invariant of one-ended finitely presented groups. Examples of

groups whose Cayley graphs have *dead-ends* (i.e. end-depth functions strictly larger than $x + c$ for any c) were obtained in [6]. Our second result shows that the (rough) equivalence class of the end-depth is not meaningful:

PROPOSITION 3.2. *Every finitely generated one-ended group has linear end-depth. More precisely we have the inequality*

$$V_0(X_G)(r) \le 2r \quad \text{for large enough } r,$$

where X_G denotes the Cayley graph associated to a finite generating set of the group G.

We postpone the proof of this proposition to the end of this section.

3.2. End of the proof of Theorem 1.6. Consider the amalgamated product $G = G_1 *_H G_2$. Let X_1 and X_2 be the standard 2-complexes associated to some finite presentations of G_1 and G_2, respectively. Let S_H be a finite set of generators of H which are represented by a wedge of loops Y in both X_1 and X_2. The space X obtained by attaching X_1 and X_2 along Y has fundamental group G. Let C_H be the Cayley graph of H corresponding to the generators S_H. The image of $\widetilde{Y}$ in $\widetilde{X}_i$ is then homeomorphic to C_H. Furthermore the universal covering $\widetilde{X}$ is constructed from coset copies of the universal coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ which are attached along copies of C_H.

We consider a metric ball $B(r)$ of radius r in $\widetilde{X}$ centered at a fixed point. By compactness, $B(r)$ intersects only finitely many copies of $\widetilde{X}_1$ and $\widetilde{X}_2$. Since $\widetilde{X}_1$ and $\widetilde{X}_2$ have linear sci, there exists a constant c such that any loop lying in one copy of either $\widetilde{X}_1$ or $\widetilde{X}_2$ which is outside $B(cr)$ is contractible by a nullhomotopy outside $B(r)$.

Since the one-ended group H has linear end-depth by Proposition 3.2, one can find a constant c_1 such that any two points of a copy of C_H lying outside $B(c_1 r)$ can be connected by a path within that copy of C_H not intersecting $B(cr)$.

The proof that any loop of $\widetilde{X}$ which lies outside $B(c_1 r)$ bounds a disk outside $B(r)$ is now standard, following [14]. Any edge loop L starting at $g \in G$ can be written as a word $ga_1 \cdots a_n$, with $a_i \in G_1$ when i is odd and $a_i \in G_2$ when i is even, such that the equality $a_1 \cdots a_n = 1$ holds in G. The structure theorem for amalgamated products implies that there exists some i such that $a_i \in H$ (see [18]). Thus the edge subpath l corresponding to the element $a_i \in H$ starts and ends in the same copy of C_H.

We will show that l can be homotoped in $\widetilde{X}$ rel end points into this copy of C_H. As L lies outside $B(c_1 r)$, the end points of l are outside $B(c_1 r)$ and by the above argument they can be connected by some path p lying within the same copy of C_H and which does not intersect $B(cr)$. The resulting loop $l \cup p$ obtained by gluing together l and p at their common end points is

therefore contained in one copy of either $\widetilde{X}_1$ or $\widetilde{X}_2$. Moreover, $l \cup p$ lies in the complement of $B(cr)$. By hypothesis, G_i have linear sci, and thus $l \cup p$ can be contracted off $B(r)$. This establishes the claim. The word associated to the path p belongs to H and it can be absorbed into a_{i-1}. Thus we obtain a free homotopy of L outside $B(r)$ to a loop L' starting at g which corresponds to a word strictly shorter than that of L. Then, by induction on n, we can decrease the length n until the resulting loop has $n = 1$. This proves the first part of Theorem 1.6.

In order to prove the second part let us recall the HNN construction. If H is a finitely generated subgroup of the finitely presented group G_1, and $f : H \to G_1$ is a monomorphism from H into G_1, we set $K = f(H)$. Suppose that H is generated by $a_1, \ldots, a_n$ and denote by c_i the generators $f(a_1), \ldots, f(a_n)$ of K. Let

$$\langle b_1, \ldots, b_m, a_1, \ldots, a_n, c_1, \ldots, c_n \mid p_1 = 1, \ldots, p_k = 1 \rangle$$

be a presentation for G_1. Then the HNN-extension $G = G_1 *_H$ of G_1 by f has the presentation

$$\langle b_1, \ldots, b_m, a_1, \ldots, a_n, c_1, \ldots, c_n, t \mid p_i = 1 \text{ for } i \leq k, \ c_j = t^{-1} a_j t \text{ for } j \leq n \rangle.$$

Consider the 2-complex X_1 associated to the given presentation of G_1. It contains two wedges of circles Y_H, Y_K associated to finite sets of generators of H and K. Consider the space X obtained from a copy of X_1 and a copy of $Y_H \times [0,1]$ where $Y_H \times \{0\}$ is identified with the copy of Y_H in X_1 and $Y_H \times \{1\}$ is identified with the copy of Y_K in X_1 by means of f. The universal covering space $\widetilde{X}$ of X can be constructed from coset copies of $\widetilde{X}_1$ and $C_H \times [0,1]$, where C_H denotes the Cayley graph of H. As above, C_H is the image of $\widetilde{Y}_H$ inside $\widetilde{X}_1$.

As in the case of an amalgamated product above, the key tool is Britton's lemma giving the structure of an HNN-extension which we state as follows. Suppose we have the equality $g_0 t^{i_1} g_1 t^{i_2} \cdots t^{i_n} g_n = 1$ in G, where $g_k \in G_1$. Then, for some k, either $i_k > 0$, $i_{k+1} < 0$, and g_k is in K, or else $i_k < 0$, $i_{k+1} > 0$, and g_k is in H.

We denote by $B(r)$ the metric ball centered at the identity in $\widetilde{X}$. Since each copy of $\widetilde{X}_1$ has linear sci, there is c such that any loop in $\widetilde{X}_1$ outside the metric ball $B(cr)$ contained in one copy of $\widetilde{X}_1$ bounds a disk not intersecting $B(r)$. The metric ball $B(cr)$ intersects only finitely many copies of $\widetilde{Y}_H \times [0,1]$. By Proposition 3.2, since H is one-ended, one can choose c large enough such that any two points of one copy of $\widetilde{Y}_H \times [0,1]$ which lie outside $B(cr)$ can be joined by a path within this copy, not intersecting $B(r)$.

Let L be an edge loop in $\widetilde{X} \smallsetminus B(c^2 r)$. This loop can be represented by a word $g_0 t^{i_1} g_1 t^{i_2} \cdots t^{i_n} g_n$, where $g_j \in G_1$ and which is equal to 1 in G. If $\sum_{j=1}^n |i_j| = 0$, then the loop is contained in one copy of $\widetilde{X}_1$, and thus

is contractible off $B(r)$, by hypothesis. When $\sum_{j=1}^{n} |i_j| > 0$, let k be the one provided by Britton's lemma in the form stated above. Then the edge path corresponding to the word $t^{\operatorname{sgn}(i_k)} g_k t^{\operatorname{sgn}(i_{k+1})}$ can be closed in either C_H or C_K by means of a path with the same end points which does not intersect $B(cr)$. Here $\operatorname{sgn}(i)$ denotes the sign of the non-zero i. We obtain a loop lying in a copy of $\widetilde{X}_1$ outside $B(cr)$, which can therefore be contracted outside $B(r)$. Thus the loop L is homotopic outside $B(r)$ to a new loop for which the quantity $\sum_{j=1}^{n} |i_j|$ dropped off by two units. The claim follows by induction.

REMARK 3.3. If G_i are one-ended sci and H is finitely generated multi-ended, then $G_1 *_H G_2$ is one-ended but *not* sci according to Jackson [14].

3.3. Proof of Proposition 3.2. The first step is the following lemma:

LEMMA 3.4. *In a homogeneous locally finite one-ended graph, through any point p passes a discrete geodesic, i.e. an isometrically embedded copy of the integers.*

Proof. Since the graph is unbounded, for any $n \in \mathbb{N}$ there exist two vertices at distance $2n$, joined by a geodesic segment $u_n, u_{n-1}, \ldots, u_{-n}$. By homogeneity, we can choose as u_0 a fixed base point $u_0 = x_0$. Now, this is true for any $n \in \mathbb{N}$, and since the graph is locally finite, there exists, by a compactness argument (e.g. diagonal extraction), the desired geodesic. ∎

Now, Proposition 3.2 follows from the following result:

PROPOSITION 3.5. *Let X be a graph as before. Let $r \in \mathbb{N}$ and K be a finite subset of X whose diameter is at most $2r$. Denote by C a connected component of $X \smallsetminus K$. Then for any point x in C, we have the following alternative: either*

- *x belongs to a geodesic ray (i.e. an embedded copy of the natural numbers) of X within C (and this in particular implies that C is infinite), or else*
- *the distance from x to K is at most r, and C is bounded.*

Proof. Let x be a point of C. Then, by Lemma 3.4, there exists a discrete geodesic (u_n) with $n \in \mathbb{Z}$ such that $u_0 = x$. If x does not belong to any geodesic ray contained in C, then one can find n and $m > 0$ (both minimal) such that u_n and u_{-m} belong to K. Since the diameter of K is by hypothesis $\leq 2r$, one has $m + n \leq 2r$. This means that the distance $d(x, K)$ from x to K is $\min\{m, n\} \leq (m + n)/2$. Hence x is within r of K. ∎

End of the proof of Proposition 3.2. Whenever K is a ball $B(r)$ of radius r centered at the neutral element of the Cayley graph of the group G, Proposition 3.5 implies that any bounded connected component of $X_G \smallsetminus B(r)$ is

included in the ball $B(2r)$ having the same center and radius $2r$. In particular, $V_0(r) \leq 2r$. ∎

An alternative proof of Proposition 3.2. Suppose that there is a positive integer $r \geq 2$ such that $V_0(r) > 2r$. Then there are a bounded connected component A of $X_G \setminus B(r)$ and $a \in A$ such that $d(a, B(r)) > r$. As G is one-ended, there is an unbounded connected component C of $X_G \setminus B(r)$. Consider the action of a on X_G by multiplication. Since $d(a, B(r)) > r$, clearly $aB(r) = B(a, r) \subset A$, and there are $x \in A$ and $y \in C$ such that $ax, ay \in B(r)$. Here $B(a, r)$ denotes the metric ball of radius r centered at a. Therefore, there is a path γ in $B(r)$ that joins ax to ay. Then $a^{-1}\gamma$ is a path that joins an element of A to an element of C, so it must pass through $B(r)$. Thus, there is w on γ so that $a^{-1}w \in B(r)$. This however implies that $w \in B(r) \cap aB(r)$, which is a contradiction. This proves that $V_0(r) \leq 2r$, and hence the end-depth of G is linear. ∎

4. Proof of Theorem 1.7. Let G be a connected, semisimple Lie group with trivial center and without compact factors. Unlike uniform lattices, non-uniform lattices Γ in G are not quasi-isometric to the symmetric space $X = G/K$ since they do not act cocompactly on X. But one can consider the following construction: chop off every cusp of the quotient X/Γ and look at the lifts of each cusp to X, giving a Γ-equivariant union of horoballs in X. These horoballs are not disjoint in general; they can be made disjoint by cutting the cusps far enough out precisely when Γ has $\mathbb{Q}$-rank one. The resulting space is called the *neutered space* X_0 associated to Γ, and Γ acts cocompactly on it. The natural metric on X_0 is the path metric induced from X, given by the infimal length in X of paths contained in X_0 that join the two points. Then Γ endowed with the word metric is quasi-isometric to X_0 endowed with the path metric. However, sometimes the path metric on X_0 might be distorted with respect to the original metric on X. In order to circumvent this difficulty we first only consider higher rank groups.

Proof of Theorem 1.7. Since G has higher rank, a result due to Lubotzky, Mozes, and Raghunathan (see [17]) states that the embedding of Γ endowed with the word metric into G endowed with a left invariant metric is Lipschitz and hence a quasi-isometric embedding. The projection $G \to G/K$ is a quasi-isometry and hence Γ is quasi-isometric to an orbit $\Gamma \cdot x_0 \subset X$ endowed with the restriction of the Riemannian metric d_X on X. Finally the embedding of an orbit of Γ into the neutered space X_0 is a quasi-isometry when we consider the metric $d_X|_{X_0}$ on X_0.

By the quasi-isometry invariance of the sci growth, it will be sufficient to prove that X_0 endowed with the metric $d_X|_{X_0}$ has a linear V_{X_0}. The metric

balls $B_{(X_0, d_X|_{X_0})}(x_0, r)$ of radius r centered at $x_0 \in X_0$ for this non-geodesic metric are easy to describe, namely

$$B_{(X_0, d_X|_{X_0})}(x_0, r) = B_{(X, d_X)}(x_0, r) \cap X_0$$

in terms of the Riemannian metric balls $B_{(X, d_X)}(x_0, r)$.

Now, the neutered space X_0 is obtained from X by removing a collection of disjoint horoballs, as the $\mathbb{Q}$-rank of Γ is at least 2. Then any ball $B_{(X, d_X)}(x_0, r)$ of X intersects only finitely many such horoballs.

This implies that the metric sphere $S_{(X_0, d_X|_{X_0})}(x_0, r) \subset \partial B_{(X_0, d_X|_{X_0})}(x_0, r)$ is obtained from the usual metric sphere $\partial B_{(X, d_X)}(x_0, r)$ in X by removing from it the intersection with a disjoint union of finitely many horoballs.

We need now a lemma which explains the geometry of such intersections:

LEMMA 4.1. *Let X be a proper* CAT(0) *manifold, H be a horoball, and B be a sphere of X. If the center c of B does not belong to H, then $B \cap H$ is convex (i.e. topologically a ball).*

Proof. Let $f_c(x) = d(x, c)$ be the distance function to a fixed point $c \notin H$. Then f_c restricted to H has only a critical point in H, namely the projection $p(c)$ of c on H, where it achieves a non-degenerate minimum. Since f_c is proper, the level sets on H retract onto $p(c)$. ∎

From Lemma 4.1 we derive that the metric spheres in $(X_0, d_X|_{X_0})$ are obtained from S^{n-1} by removing finitely many disjoint disks D^{n-1}. This means that, whenever the dimension n of X is $n \geq 4$, the metric spheres in X_0 are simply connected. It follows that $V_{(X_0, d_X|_{X_0})}(r) = r$ is linear and hence Γ has linear sci.

For the second part of Theorem 1.7 consider a non-uniform lattice Γ in $\mathrm{SO}(n, 1)$. The non-uniform lattice Γ acts properly and cocompactly by isometries on $X_0 = \mathbb{H}^n \smallsetminus \mathcal{F}$ where $\mathcal{F}$ is a finite union of disjoint open horoballs. The result does not follow from Lemma 4.1, as the metric on this truncated hyperbolic metric space is the path metric, which is exponentially distorted. Nevertheless this space is CAT(0) (by [5, Cor. 11.28, p. 362] and [24]). Metric balls are therefore homeomorphic to balls and their boundaries are spheres.

In order to understand the topology of the metric spheres it suffices to consider a neighborhood of one horoball H. Given $c \in X_0$ consider the cone in $\mathbb{H}^n$ with vertex c which is tangent to the horoball H along an equidistant $(n-1)$-sphere $S^{n-1}(c) \subset \partial H$. If p belongs to the visible n-disk bounded by S^{n-1} on ∂H, then the geodesic segments joining p and c for the hyperbolic metric $d_{\mathbb{H}^n}$ and the path metric on X_0 coincide. When $p \in \partial H$ is outside the visible disk then a geodesic segment in the path metric consists of a spherical segment pq joining p to $q \in S^{n-1}(c)$ followed by a geodesic segment qc. It follows that metric spheres in the path metric are obtained from a sphere by

deleting a number of disjoint disks corresponding to visible disks at distance smaller than the radius. For $n \geq 4$ these are simply connected, and this shows that X_0 with its path metric has linear sci. ■

5. Other classes of groups with linear sci. Recall that a *Coxeter group* is a group W with a presentation of the following form:

$$\langle s_1, \ldots, s_n \mid s_i^2 = 1 \text{ for } i \in \{1, \ldots, n\}, (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $i < j$ ranges over some subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ and $m_{ij} \geq 2$. Let W be a Coxeter group.

PROPOSITION 5.1. *Coxeter groups which are sci have linear sci.*

Proof. The Davis complex D_W (see [7]) of a finitely generated Coxeter group W is a CAT(0) cell complex on which W acts cellularly, properly, and with finite quotient. The links of the vertices of D_W are all isomorphic to a fixed finite simplicial complex L, where L can be described combinatorially in terms of subsets of the generating set of W.

It has been proved in [8] that a Coxeter group is sci if and only if its *nerve L* and all its *punctured links $L - \sigma$* are simply connected (where σ is any simplex of L). The boundary of a metric ball in D_W is a connected sum of various punctured links $L - \sigma$, and hence it is simply connected.

Now any loop outside the metric ball of radius r can be contracted onto the boundary of the metric ball and there contracted to a point. This implies that $V_{D_W}(r) = r$.

The action of the Coxeter group on the Davis complex is not free but has finite stabilizers. Moreover there exists a finite index subgroup which acts freely on the Davis complex. This finite index subgroup is still sci and quasi-isometric to D_W, and hence by the previous arguments it has linear sci. This implies that W has linear sci. ■

REMARK 5.2. The same proof shows that a sci right-angled Artin group has linear sci. More generally, Artin groups are semistable and have linear semistability (see [20]).

REMARK 5.3. The connectivity of the punctured links determines the connectivity of W at infinity. However in [9] the authors constructed a CAT(0) cell complex acted on properly and cocompactly by W whose nerve and punctured links are *not* simply connected, though W is sci. Thus the linear sci is the geometric property of groups which is closest to the "simple connectivity of large spheres".

REMARK 5.4. If $1 \to H \to G \to K \to 1$ is an exact sequence of finitely presented infinite groups where either H or K has one end, then G has a linear sci growth, by [14].

REMARK 5.5. Mihalik and Tschantz [21, 22] have proved that amalgamated products and HNN-extensions of semistable groups over arbitrary finitely generated subgroups are semistable. We do not know whether Theorem 1.6 can be extended to multi-ended subgroups and linear semistability.

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