A SHARP INTEGRAL REARRANGEMENT INEQUALITY FOR THE DYADIC MAXIMAL OPERATOR AND APPLICATIONS

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Abstract: We prove a sharp integral inequality for the dyadic maximal operator and give as an application another proof for the computation of its Bellman function of three variables.

Keywords:Bellman, Dyadic, Maximal, Rearrangement.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|du : \ x \in Q,\ Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, 2, \dots$.

It is well known that it satisfies the following weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi > \lambda\}} |\phi(u)|du, \tag{1.2}$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$.

Using this inequality it is not difficult to prove the following known as Doob's inequality

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1}\|\phi\|_p, \tag{1.3}$$

for every $p > 1$ and $\phi \in L^p(\mathbb{R}^n)$.

It is an immediate result that the weak type inequality (1.2) is best possible, while (1.3) is also sharp (see [1], [2] for general martingales and [16] for dyadic ones).

A way of studying the dyadic maximal operator is by making refinements of the above inequalities. The above inequalities hold true even in more general settings. More precisely we consider a non-atomic probability space (X, μ) equipped with a tree structure $\mathcal{T}$ and define

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi|d\mu : \ x \in I \in \mathcal{T}\right\}.$$

This research has been co-financed by the European Union and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF). Aristeia code:MaxBellman 2760, Research code:70/3/11913

Concerning (1.2) certain refinements have been done in [8] and [9] while for (1.3) the Bellman function of the dyadic maximal operator has been explicitly computed in [3]. This is given by

$$B_p(f,F)=\sup\left\{\int_X(\mathcal{M}_\mathcal{T}\phi)^pd\mu:\ \phi\geq 0,\ \int_X\phi d\mu=f,\ \int_X\phi^pd\mu=F\right\}, \tag{1.4}$$

for $p>1$ and every f and F such that $0<f^p\leq F$.

It is proved in [3] that it equals

$$B_p(f,F)=F\omega_p(f^p/F)^p,\quad\text{where}\ \ \omega_p:[0,1]\ \to\ \left[1,\frac{p}{p-1}\right]$$

denotes the inverse function H_p^{-1} of H_p, which is defined by $H_p(z)=-(p-1)z^p+pz^{p-1}$, for $z\in[1,\frac{p}{p-1}]$.

After this evaluation the second task is to find the exact value of the following function of three variables

$$B_p(f,F,L)=\sup\left\{\int_X\max(\mathcal{M}_\mathcal{T}\phi,L)^pd\mu:\ \phi\geq 0,\ \int_X\phi d\mu=f,\ \int_X\phi^pd\mu=F\right\}, \tag{1.5}$$

for $p>1$, $0<f^p\leq F$ and $L\geq f$.

It turns out that

$$B_p(f,F,L)=\begin{cases} F\omega_p\Big(\dfrac{pL^{p-1}f-(p-1)L^p}{F}\Big)^p, & \text{if}\quad L<\dfrac{p}{p-1}f\\ L^p+\Big(\dfrac{p}{p-1}\Big)^p\Big(F-f^p\Big) & \text{if}\quad L>\dfrac{p}{p-1}f. \end{cases} \tag{1.6}$$

For this evaluation the author in [3] used the result for (1.4) on suitable subsets of X and after several calculus arguments he was able to provide a proof of (1.6).

The Bellman functions have been studied also in [4]. There a more general Bellman function has been computed, namely

$$\begin{aligned} T_{p,G,H}(f,F,k)=\sup\Big\{\int_K G(\mathcal{M}_\mathcal{T}\phi)d\mu:\ \phi\geq 0,\ \int_X\phi d\mu=f,\ \int H(\phi)d\mu=F,\\ K\ \text{measurable subset of}\ X\ \text{with}\ \mu(K)=k\Big\} \end{aligned} \tag{1.7}$$

for suitable convex, non-negative, increasing functions G and H. The approach used in [4] is by proving that $T_{p,G,H}(f,F,k)$ equals

$$\begin{aligned} S_{p,G,H}(f,F,k)=\sup\Big\{\int_0^k G\Big(\frac{1}{t}\int_0^t g\Big)dt:\ g:(0,1]\ \to\ \mathbb{R}^+\ \text{non-increasing, continuous}\\ \text{with}\ \int_0^1 g(u)du=f,\ \int_0^1 H(g)dt=F\Big\}. \end{aligned}$$

The second step then is to evaluate $S_{p,G,H}(f,F,k)$, which in general is a difficult task. Concerning the first step ($T_{p,G,H}=S_{p,G,H}$) the following equality has been proved in

[10] stated as

Theorem A. *If $g,h:(0,1]\to\mathbb{R}^+$ are non-increasing integrable functions and $G:[0,+\infty)\to[0,+\infty)$ is non-decreasing, then the following is true*

$$\sup\left\{\int_K G[(\mathcal{M}_{\mathcal{T}}\phi)^*]h(t)dt,\ \phi^*=g,\ K \textit{ measurable subset of } (0,1] \textit{ with } |K|=k\right\}$$
$$=\int_0^k G\left(\frac{1}{t}\int_0^t g(u)du\right)h(t)dt.$$

This can be viewd as a symmetrization principle that immediately yields the equality $T_{p,G,H}=S_{p,G,H}$.

In this paper our aim is to find another proof of (1.6) by using a variant of Theorem A. More precisely we will prove the following

Theorem 1. *The following equality is true*

$$\sup\left\{\int_K G_1(\mathcal{M}_{\mathcal{T}}\phi)G_2(\phi)d\mu:\phi^*=g,\ K \textit{ measurable subset of}\right.$$
$$\left. X \textit{ with } \mu(K)=k\right\}=\int_0^k G_1\left(\frac{1}{t}\int_0^t g\right)G_2(g(t))dt,$$

where $G_i:[0,+\infty)\to[0,+\infty)$ are increasing functions for $i=1,2$, while $g:(0,1]\to\mathbb{R}$ is non-increasing.

This theorem and some extra effort will enable us to provide a simpler proof of (1.6). We also remark that there are several problems in Harmonic Analysis were Bellman functions arise. Such problems (including the dyadic Carleson imbedding theorem and weighted inequalities) are described in [7] (see also [5], [6]) and also connections to Stochastic Optimal Control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second-order PDEs. The exact evaluation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding Harmonic Analysis problem. Until now several Bellman functions have been computed (see [1], [2], [3], [5], [12], [13], [14], [15]). The exact evaluation of (1.4) has been also given in [11] by L. Slavin, A. Stokolos and V. Vasyunin which linked the computation of it to solving certain PDEs of the Monge-Ampère type and in this way they obtained an alternative proof of the results in [3] for the Bellman functions related to the dyadic maximal operator.

The paper is organized as follows. In Section 2 we give some preliminaries needed for use in the subsequent sections. In Section 3 we prove Theorem 1 while in Section 4 we give a proof that the right side of (1.6) is an upper bound of the quantity: $\int\limits_X \max(\mathcal{M}_{\mathcal{T}}\phi,L)^p d\mu$. At last in Section 5 we prove the sharpness of the above mentioned result.

2. Preliminaries

Let (X, μ) be a non-atomic probability measure space.

Definition 2.1. *A set $\mathcal{T}$ of measurable subsets of X will be called a tree if it satisfies the following conditions*

i) *$X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.*
ii) *For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that*
   (a) *the elements of $C(I)$ are pairwise disjoint subsets of I*
   (b) *$I = \cup C(I)$.*
iii) *$\mathcal{T} = \bigcup\limits_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup\limits_{I \in \mathcal{T}_{(m)}} C(I)$.*
iv) *We have that $\lim\limits_{m \to \infty} \sup\limits_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$.*

Examples of trees are given in [3]. The most known is the one given by the family of all dyadic subcubes of $[0,1]^n$. The following has been proved in [3].

Lemma 2.1. *For every $I \in \mathcal{T}$ and every a such that $0 < a < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of disjoint subsets of I such that*

$$\mu\Bigg(\bigcup_{J \in \mathcal{F}(I)} J \Bigg) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1-a)\mu(I).$$

We will also need the following fact obtained in [10].

Lemma 2.2. *Let $\phi : (X, \mu) \to \mathbb{R}^+$ and $(A_j)_j$ a measurable partition of X such that $\mu(A_j) > 0 \ \forall \ j$. Then if $\int\limits_X \phi d\mu = f$ there exists a rearrangement of ϕ, say h $(h^* = \phi^*)$ such that $\dfrac{1}{\mu(A_j)} \int\limits_{A_j} h d\mu = f$, for every j.*

Here by ϕ^* we mean the decreasing rearrangement of ϕ defined by

$$\phi^*(t) = sup_{e \subset X, |e| = t} [inf_{x \in e} |\phi(x)|], t \in (0,1].$$

Now given a tree on (X, μ) we define the associated dyadic maximal operator as follows

$$\mathcal{M}_{\mathcal{T}} \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : \ x \in I \in \mathcal{T} \right\},$$

for every $\phi \in L^1(X, \mu)$.

We will also need the following well known (see[17]).

Lemma 2.3. *Let $\phi_1, \phi_2 : X \to \mathbb{R}^+$ be μ-measurable functions. Then the following inequality is always true:*

$$\int_X \phi_1(x) \phi_2(x) d\mu(x) \leq \int_0^1 \phi_1^*(t) \cdot \phi_2^*(t) dt$$

where ϕ_i^* is decreasing rearrangement of ϕ_i.

3. The rearrangement inequality

We prove first the following

Lemma 3.1. *With the notation of Theorem 1 the following inequality holds*

$$\int_K G_1(\mathcal{M}_\mathcal{T}\phi)G_2(\phi)d\mu \leq \int_0^k G_1\left(\frac{1}{t}\int_0^t g\right)G_2(g(t))dt$$

Proof. Following [10] we set

$$I = \int_K G_1(\mathcal{M}_\mathcal{T}\phi)G_2(\phi)d\mu.$$

Then by using Lemma 2.3 we have that:

$$I \leq \int_0^k [G_1(\mathcal{M}_\mathcal{T}\phi)/K]^* \cdot [G_2(\phi)/K]^* dt.$$

Since $K \subseteq X$ we have that

$$[G_1(\mathcal{M}_\mathcal{T}\phi)/K]^*(t) \leq [G_1(\mathcal{M}_\mathcal{T}\phi)]^*(t) \quad \text{and}$$
$$[G_2(\phi)/K]^*(t) \leq [G_2(\phi)]^*(t), \qquad \text{for any } \ t \in (0,k].$$

On the other hand, G_1 and G_2 are increasing functions, therefore

$$[G_1(\mathcal{M}_\mathcal{T}\phi)]^* = G_1[(\mathcal{M}_\mathcal{T}\phi)^*] \quad \text{and}$$
$$[G_2(\phi)]^* = G_2(\phi^*),$$

almost everywhere with respect to the Lesbesgue measure on $(0,k]$. Thus

$$I \leq \int_0^k G_1[(\mathcal{M}_\mathcal{T}\phi)^*(t)] \cdot G_2(g(t))dt = II.$$

The last integral now equals

$$II = \int_0^k G_1[(\mathcal{M}_\mathcal{T}\phi)^*(t)]dv_2(t),$$

where v_2 is the Borel measure defined on $(0,k]$ by

$$v_2(A) = \int_A G_2(g(u))du.$$

Then we have that

$$II = \int_{\lambda=0}^{+\infty} v_2(\{t \in (0,k] : (\mathcal{M}_\mathcal{T}\phi)^*(t) \geq \lambda\})d\,G_1(\lambda) = III + IV, \quad \text{where}$$

$$III = \int_0^f v_2((0,k])d\,G_1(\lambda) = v_2((0,k])[G_1(f) - G_1(0)] \quad \text{and}$$

$$IV = \int_{\lambda=f}^{+\infty} v_2(\{t \in (0,k] : (\mathcal{M}_\mathcal{T}\phi)^*(t) \geq \lambda\})dG_1(\lambda). \tag{3.1}$$

Now we will prove that if we set

$$A_\lambda = \{t \in (0,k] : (\mathcal{M}_\mathcal{T}\phi)^*(t) \geq \lambda\} \quad \text{and}$$

$$\Omega_\lambda = \left\{ t \in (0,k] : \frac{1}{t}\int_0^t g \geq \lambda \right\},$$

then $A_\lambda \subseteq \Omega_\lambda$, for any $\lambda > f$. Fix such a λ.

Since A_λ and Ω_λ are defined in terms of non-increasing functions on $(0,k]$ we must have that

$$A_\lambda = (0, |A_\lambda|], \quad \text{and} \quad \Omega_\lambda = (0, |\Omega_\lambda|],$$

that is they must be intervals with 0 being their common left end-point. Thus in order to prove that $A_\lambda \subseteq \Omega_\lambda$ we just need to show that $|A_\lambda| \leq |\Omega_\lambda|$.

For our fixed λ we have that there exists $\beta(\lambda) \in (0,1]$ such that $\frac{1}{\beta(\lambda)}\int_0^{\beta(\lambda)} g(u)du = \lambda$. It's existence is guaranteed by the fact that $\lambda > f = \int_0^1 g(u)du$. In fact, we can suppose without loss of generality that $g(0^+) = +\infty$, otherwise we work on $\lambda \in (f, \|g\|_\infty]$. Notice that if $\|g\|_\infty = A$, then $\mathcal{M}_\mathcal{T}\phi \leq A$ μ-a.e. on X.

By the definition of Ω_λ and $\beta(\lambda)$ it follows that $\Omega_\lambda = (0, \min(\beta(\lambda), k)]$. Also note that $|A_\lambda| \leq k$. Therefore it suffices to prove that $|A_\lambda| \leq \beta(\lambda)$. But

$$A_\lambda \subseteq \{t \in (0,1] : (\mathcal{M}_\mathcal{T}\phi)^*(t) \geq \lambda\} \Rightarrow |A_\lambda| \leq |\{t \in (0,1] : (\mathcal{M}_\mathcal{T}\phi)^*(t) \geq \lambda\}| = \mu(E_\lambda),$$

where E_λ is defined by

$$E_\lambda = \{x \in X : (\mathcal{M}_\mathcal{T}\phi)(x) \geq \lambda\}.$$

There exists a pairwise disjoint family of elements of $\mathcal{T}$, $(I_j)_j$, such that

$$\frac{1}{\mu(I_j)}\int_{I_j} \phi d\mu \geq \lambda \ \text{ and } \ E_\lambda = \cup I_j. \tag{3.2}$$

In fact we just need to consider the family $(I_j)_j$ of elements of $\mathcal{T}$, maximal under the above integral condition.

By (3.3) we have that $\int_{I_j} \phi d\mu \geq \lambda\mu(I_j)$, for any j, and so summing the above inequalities with respect to j, we conclude that

$$\int_{E_\lambda} \phi d\mu \geq \lambda\mu(E_\lambda) \ \text{ or that } \ \frac{1}{\mu(E_\lambda)}\int_{E_\lambda} \phi d\mu \geq \lambda.$$

On the other hand $\beta(\lambda)$ is defined by the equation:

$$\frac{1}{\beta(\lambda)}\int_0^{\beta(\lambda)} g(u)du = \lambda.$$

So we have have the following inequalities

$$\frac{1}{\mu(E_\lambda)}\int_0^{\mu(E_\lambda)} g(u)du \geq \frac{1}{\mu(E_\lambda)}\int_{E_\lambda} \phi d\mu \geq \lambda = \frac{1}{\beta(\lambda)}\int_0^{\beta(\lambda)} g(u)du.$$

implying that $\mu(E_\lambda) \leq \beta(\lambda)$, since g is non-increasing. Then because of the inequality $|A_\lambda| \leq \mu(E_\lambda)$ we have $|A_\lambda| \leq |\Omega_\lambda|$. By the above we find that

$$A_\lambda \subseteq \Omega_\lambda \Rightarrow v_2(A_\lambda) \leq v_2(\Omega_\lambda).$$

Now using (3.1) we get

$$IV \leq \int_{\lambda=f}^{+\infty} v_2\Big(\Big\{t \in (0,k] : \frac{1}{t}\int_0^t g \geq \lambda\Big\}\Big) dG_1(\lambda), \quad \text{thus}$$

$$\begin{aligned} I &\leq \int_{\lambda=0}^{+\infty} v_2\Big(\Big\{t \in (0,k] : \frac{1}{t}\int_0^t g \geq \lambda\Big\}\Big) dG_1(\lambda) \\ &= \int_0^k G_1\Big(\frac{1}{t}\int_0^t g\Big) dv_2(t) = \int_0^k G_1\Big(\frac{1}{t}\int_0^t g\Big) G_2(g(t)) dt \end{aligned}$$

by the definition of v_2. This completes the proof of Lemma 3.1. □

We now proceed to the

Proof of Theorem 1: First suppose that $k = 1$. Let $g : (0,1] \to \mathbb{R}^+$ be a non-increasing function. We are going to construct a family $(\phi_a)_{a\in(0,1)}$ of functions defined on (X,μ), each having g as it's decreasing rearrangement ($\phi_a^* = g$), such that

$$\limsup_{a \to 0^+} \int_X G_1(\mathcal{M}_{\mathcal{T}}\phi_a) G_2(\phi_a) d\mu \geq \int_0^1 G_1\Big(\frac{1}{t}\int_0^t g\Big) G_2(g(t)) dt.$$

Following [10] we let $a \in (0,1)$. Using Lemma 2.1 we choose for every $I \in \mathcal{T}$ a family $\mathcal{F}(I) \subseteq \mathcal{T}$ of disjoint subsets of I such that

$$\sum_{J\in\mathcal{F}(I)} \mu(J) = (1-a)\mu(I). \tag{3.3}$$

Define $S = S_a$ by induction to be the smallest subset of $\mathcal{T}$ for which $X \in S$ and for every $I \in S$, $\mathcal{F}(I) \subseteq S$. We write for $I \in S$, $A_I = I \setminus \bigcup_{J\in\mathcal{F}(I)} J$. Then if $a_I = \mu(A_I)$ we have because of (3.3) that $a_I = a\mu(I)$. It is also clear that

$$S_a = \bigcup_{m\geq 0} S_{a,(m)}, \quad \text{where} \quad S_{a,(0)} = \{X\} \quad \text{and} \quad S_{a,(m+1)} = \bigcup_{I\in S_{a,(m)}} \mathcal{F}(I).$$

We define also for $I \in S_a$, $\operatorname{rank}(I) = r(I)$ to be the unique integer m such that $I \in S_{a,(m)}$. Additionally, we define for every $I \in S_a$ with $r(I) = m$

$$\gamma(I) = \gamma_m = \frac{1}{a(1-a)^m}\int_{(1-a)^{m+1}}^{(1-a)^m} g(u)du.$$

and

$$b_m(I) = \sum_{\substack{S\ni J\subseteq I \\ r(J)=r(I)+m}} \mu(J).$$

We easily then see inductively that

$$b_m(I) = (1-a)^m\mu(I).$$

It is also clear that for every $I \in S_a$

$$I = \bigcup_{S_a\ni J\subseteq I} A_J.$$

At last we define for every m the measurable subset of X, $S_m = \bigcup_{I\in S_{a,(m)}} I$. Now for each $m \geq 0$ we choose $\tau_a^{(m)} : S_m \setminus S_{m+1} \to \mathbb{R}$ such that

$$\left[\tau_a^{(m)}\right]^* = \Big(g/\big((1-a)^{m+1}, (1-a)^m\big]\Big)^*.$$

This is possible since $\mu(S_m \setminus S_{m+1}) = \mu(S_m) - \mu(S_{m+1}) = b_m(X) - b_{m+1}(X) = (1-a)^m - (1-a)^{m+1} = a(1-a)^m$. It is obvious that $S_m \setminus S_{m+1} = \bigcup_{I\in S_{a,(m)}} A_I$ and that

$$\int_{S_m\setminus S_{m+1}} \tau_a^{(m)} d\mu = \int_{(1-a)^{m+1}}^{(1-a)^m} g(u)du \Rightarrow \frac{1}{\mu(S_m \setminus S_{m+1})} \int_{S_m\setminus S_{m+1}} \tau_a^{(m)} d\mu = \gamma_m.$$

Define $\tau_a : X \to \mathbb{R}^+$ by $\tau_a/(S_m \setminus S_{m+1}) := \tau_a^{(m)}$, $m \geq 0$. Using Lemma 2.2 we see that there exists a rearrangement of $\tau_a^{(m)}$, called $\phi_a^{(m)}$, for which $\frac{1}{a_I}\int_{A_I} \phi_a^{(m)} = \gamma_m$, for every $I \in S_{a,(m)}$. We define $\phi_a : X \to \mathbb{R}^+$ by $\phi_a(x) = \phi_a^{(m)}(x)$, for $x \in S_m \setminus S_{m+1}$. Clearly $\phi_a^* = g$.

Let now $I \in S_{a,(m)}$. Then

$$\begin{aligned}
&\frac{1}{\mu(I)}\int_I \phi_a d\mu \\
&= \frac{1}{\mu(I)} \sum_{S_a\ni J\subseteq I} \int_{A_J} \phi_a d\mu \\
&= \frac{1}{\mu(I)} \sum_{\ell\geq 0} \sum_{\substack{S_a\ni J\subseteq I \\ r(J)=r(I)+\ell}} \int_{A_J} \phi_a d\mu \\
&= \frac{1}{\mu(I)} \sum_{\ell\geq 0} \sum_{S_a\ni J\subseteq I} \gamma_{m+\ell} a_J \\
&= \frac{1}{\mu(I)} \sum_{\ell\geq 0} \sum_{S_a\ni J\subseteq I} a\mu(J) \frac{1}{a(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u)du \\
&= \frac{1}{\mu(I)} \sum_{\ell\geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)_{m+\ell}} g(u)du \cdot \sum_{\substack{S_a\ni J\subseteq I \\ r(J)=m+\ell}} \mu(J) \\
&= \frac{1}{\mu(I)} \sum_{\ell\geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u)du \cdot b_\ell(I) \\
&= \frac{1}{(1-a)^m} \sum_{\ell\geq 0} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u)du
\end{aligned}$$

$$= \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u)du. \tag{3.4}$$

Now for $x \in S_m \setminus S_{m+1}$, there exists $I \in S_{a,(m)}$ such that $x \in I$ so

$$\mathcal{M}_{\mathcal{T}}(\phi_a)(x) \geq \frac{1}{\mu(I)} \int_I \phi_a d\mu = \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u)du =: \theta_m, \tag{3.5}$$

Then for each $a \in (0,1)$ we have that

$$\begin{aligned}\int_X G_1(\mathcal{M}_{\mathcal{T}}\phi_a)G_2(\phi_a)d\mu &= \sum_{\ell\geq 0} \int_{S_\ell \setminus S_{\ell+1}} G_1(\mathcal{M}_{\mathcal{T}}\phi_a)G_2(\phi_a)d\mu \geq \text{(due to (3.5))} \\ &\geq \sum_{\ell \geq 0} G_1(\theta_\ell) \int_{S_\ell\setminus S_{\ell+1}} G_2(\phi_a)d\mu. \end{aligned} \tag{3.6}$$

By the construction now of ϕ_a we note that

$$\Big(\phi_a/S_\ell \setminus S_{\ell+1}\Big)^* = \Big(g/((1-a)^{\ell+1}, (1-a)^\ell]\Big)^*,$$

so (3.6) becomes

$$\begin{aligned}\int_X G(\mathcal{M}_{\mathcal{T}}\phi_a)G_2(\phi_a)d\mu &\geq \sum_{\ell\geq 0} G_1\bigg(\frac{1}{(1-a)^\ell}\int_0^{(1-a)^\ell} g(u)du\bigg)\cdot \int_{(1-a)^{\ell+1}}^{(1-a)^\ell} G_2(g(u))du \\ &\geq \sum_{\ell\geq 0} G_1\bigg(\frac{1}{(1-a)^\ell}\int_0^{(1-a)^\ell} g(u)du\bigg) a(1-a)^\ell G_2(g((1-a)^\ell)) \\ &= \sum_{\ell\geq 0} G_1\bigg(\frac{1}{(1-a)^\ell}\int_0^{(1-a)^\ell} g(u)du\bigg) G_2(g((1-a)^\ell))|((1-a)^{\ell+1},(1-a)^\ell]|.\end{aligned} \tag{3.7}$$

The sum in (3.7) is a Riemman sum of the integral $\int\limits_0^1 G_1\Big(\frac{1}{t}\int\limits_0^t g\Big)G_2(g(t))dt$, so as $a \to 0^+$, we see that we have the needed inequality. The general case of the sharpness of Lemma 3.1 for any k can be proved along the same lines, integrating $G_1(\mathcal{M}_{\mathcal{T}}\phi_a)\cdot G_2(\phi_a)$ on S_{m_a} for each a, where $m_a \in \mathbb{N}$ is such that $(1-a)^{m_a+1} < k \leq (1-a)^{m_a}$,and thus $(1-a)^{m_a} \to k$, so by continuity reasons we have the result.

4. The Bellman function

We consider now a non-increasing function $g : (0,1] \to \mathbb{R}^+$ and the quantities

$$v_g(L) = \int\limits_{t=0}^{1} \max\left(\frac{1}{t}\int\limits_0^t g, L\right)^p dt \quad \text{and}$$
$$u_g(L) = \int\limits_{t=0}^{1} g(t)\max\left(\frac{1}{t}\int\limits_0^t g, L\right)^{p-1} dt.$$

where $L \geq f$. We will prove the following

Lemma 4.1. *With the above notation the following equality holds for every $g : (0,1] \to \mathbb{R}^+$,*

$$v_g(L) = L^p - \frac{p}{p-1} f L^{p-1} + \frac{p}{p-1} u_g(L). \tag{4.1}$$

Proof. We have that

$$\begin{aligned} v_g(L) &= \int_{\lambda=0}^{L} + \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\left|\left\{t\in(0,1] : \max\left(\frac{1}{t}\int_0^t g, L\right) \geq \lambda\right\}\right| d\lambda \\ &= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\left|\left\{t\in(0,1] : \frac{1}{t}\int_0^t g \geq \lambda\right\}\right| d\lambda. \end{aligned}$$

We consider now for each $\lambda > L \geq f$, the unique $\beta(\lambda) \in (0,1]$ such that $\frac{1}{\beta(\lambda)}\int\limits_0^{\beta(\lambda)} g(u)du = \lambda$ (we suppose that $g(0^+) = +\infty$, without loss of the generality). Therefore,

$$v_g(L) = L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1}|A_\lambda| d\lambda,$$

where

$$A_\lambda = \left\{t\in(0,1] : \frac{1}{t}\int_0^t g > \lambda\right\} = (0,\beta(\lambda)). \text{ So}$$

$$\begin{aligned} v_g(L) &= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\beta(\lambda)d\lambda \\ &= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\left(\frac{1}{\lambda}\int_0^{\beta(\lambda)} g(u)du\right)d\lambda \\ &= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-2}\left(\int_{\{u:\frac{1}{u}\int\limits_0^u g>\lambda\}} g(u)du\right)d\lambda \\ &= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-2}\left(\int_{\{u:\max\left(\frac{1}{u}\int\limits_0^u g, L\right)>\lambda\}} g(u)du\right)d\lambda \\ &= L^p + \int_0^1 g(t)\frac{p}{p-1}\left[\lambda^{p-1}\right]_{\lambda=L}^{\max\left(\frac{1}{t}\int\limits_0^t g, L\right)} dt \\ &= L^p - \frac{p}{p-1}L^{p-1}f + \frac{p}{p-1}u_g(L), \end{aligned}$$

where in the previous to the last inequality we have used Fubini's theorem. Lemma 4.1 is now proved. □

We now prove the following

Lemma 4.2. *For every f and F such that $0 < f^p \leq F$ and $L \geq f$ we have that*

$$\int_X \max(\mathcal{M}_\mathcal{T}\phi, L)^p d\mu \leq \begin{cases} F\omega_p\Big(\dfrac{pL^{p-1}f-(p-1)L^p}{F}\Big)^p F, & \text{if } L < \dfrac{p}{p-1}f \\ F^p + \Big(\dfrac{p}{p-1}\Big)^p (F-f^p), & \text{if } L \geq \dfrac{p}{p-1}f \end{cases}$$

for every ϕ such that, $\int\limits_X \phi d\mu = f$ and $\int\limits_X \phi^p d\mu = F$.

Proof. We set $I = \int\limits_X \max(\mathcal{M}_{\mathcal{T}}\phi, L)^p d\mu$. Then

$$\begin{aligned} I &= \int_{\lambda=0}^{+\infty} p\lambda^{p-1}\mu(\{x \in X : \max(\mathcal{M}_{\mathcal{T}}\phi(x), L) > \lambda\})d\lambda \\ &= \int_{\lambda=0}^{L} + \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\mu(\{x \in X : \max(\mathcal{M}_{\mathcal{T}}\phi(x), L) > \lambda\})d\lambda \\ &= II + III, \quad \text{where} \end{aligned}$$

$$II = \int_{\lambda=0}^{L} p\lambda^{p-1} d\lambda = L^p,$$

since (X, μ) is a probability space, and

$$III = \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\mu(\{x \in X : \mathcal{M}_{\mathcal{T}}\phi(x) > \lambda\})d\lambda.$$

By the weak type inequality (1.2) we obtain that

$$\begin{aligned} III &\leq \int_{\lambda=L}^{+\infty} p\lambda^{p-1}\left(\frac{1}{\lambda}\int_{\{\mathcal{M}_{\mathcal{T}}\phi>\lambda\}} \phi d\mu\right)d\lambda \\ &= \int_{\lambda=L}^{+\infty} p\lambda^{p-2}\left(\int_{\{\max(\mathcal{M}_{\mathcal{T}}\phi, L)>\lambda\}} \phi d\mu\right)d\lambda \\ &= \int_X \phi(x)\left(\int_{\lambda=L}^{\max(\mathcal{M}_{\mathcal{T}}\phi(x), L)} p\lambda^{p-2}d\lambda\right)d\mu(x) \\ &= \int_X \phi(x)\frac{p}{p-1}\left[\lambda^{p-1}\right]_{\lambda=L}^{\max(\mathcal{M}_{\mathcal{T}}\phi(x), L)} d\mu(x) \end{aligned}$$

$$= \frac{p}{p-1}\int_X \phi(x)\max(\mathcal{M}_{\mathcal{T}}\phi(x), L)^{p-1}d\mu(x) - \frac{p}{p-1}L^{p-1}f. \tag{4.2}$$

By (4.2) then

$$\begin{aligned} III &\leq \frac{p}{p-1}\left(\int_X \phi^p d\mu\right)^{1/p} \cdot \left(\int_X \max(\mathcal{M}_{\mathcal{T}}\phi, L)^p\right)^{(p-1)/p} - \frac{p}{p-1}L^{p-1}f \Rightarrow \\ I &\leq \frac{p}{p-1}F^{1/p}I^{(p-1)/p} + L^p - \frac{p}{p-1}L^{p-1}f \Rightarrow \\ \frac{I}{F} &\leq \frac{p}{p-1}\left(\frac{I}{F}\right)^{(p-1)/p} + \frac{L^p - \frac{p}{p-1}L^{p-1}f}{F} \Rightarrow \end{aligned}$$

$$\Rightarrow pw^{p-1} - (p-1)w^p \geq \frac{pL^{p-1}f - (p-1)L^p}{F},$$

where $w = \left(\frac{I}{F}\right)^{1/p}$. This gives

$$-(p-1)w^p + pw^{p-1} = H_p(w) \geq \frac{pL^{p-1}f - (p-1)L^p}{F} \tag{4.3}$$

where the function H_p is defined on $\left[1, \frac{p}{p-1}\right]$ with values on $[0, 1]$.

We consider the function $h : [f, +\infty) \to \mathbb{R}$ defined by

$$h(t) = pt^{p-1}f - (p-1)t^p, \quad t \geq f.$$

Then

$$\begin{aligned} h'(t) &= p(p-1)t^{p-2}f - p(p-1)t^{p-1} \\ &= p(p-1)(f-t)t^{p-2} < 0 \Rightarrow \ h \text{ is strictly decreasing in it's domain} \end{aligned}$$

Therefore, $h(t) \leq h(f) = f^p$ for every $t \geq f$, thus the right side of (4.3) which we denote by b, is less than $f^p/F \leq 1$.

We consider two cases

i) $b \geq 0$. Then we have that $b \in [0,1]$ and $H_p(\omega) \geq b$. If $w \leq 1$ then we must have that $I \leq F$ which gives in view of the fact that $\omega_p(b) > 1$, the inequality $I \leq F[\omega_p(b)]^p$, that is our result. We consider now the case $w > 1$. Then since $H_p : \left[1, \dfrac{p}{p-1}\right] \to [0,1]$ is strictly decreasing we have that

$$\begin{aligned} H_p(w) \geq b \Rightarrow w \leq \omega_p(b) \Rightarrow \frac{I}{F} \leq [\omega_p(b)]^p \\ \Rightarrow I \leq F\omega_p\left(\frac{pL^{p-1}f - (p-1)L^p}{F}\right)^p, \end{aligned}$$

We have proved our Lemma in the first case.

ii) We consider now the second case: $b < 0$ that is $L > L_0 = \dfrac{p}{p-1}f$. Then

$$I = \int_X \max(\mathcal{M}_\mathcal{T}\phi, L)^p d\mu = L^p + III$$

where as we have seen

$$III \leq \int_{\lambda=L}^{+\infty} p\lambda^{p-2}\left(\int_{\{\mathcal{M}_\mathcal{T}\phi > \lambda\}} \phi d\mu\right)d\lambda. \tag{4.4}$$

Since $L > L_0$ we conclude by (4.3) that

$$III \leq \int_{\lambda=L_0}^{+\infty} p\lambda^{p-2}\left(\int_{\{\mathcal{M}_\mathcal{T}\phi > \lambda\}} \phi d\mu\right)d\lambda = \int_X \max(\mathcal{M}_\mathcal{T}\phi, L_0)^p d\mu - L_0^p.$$

By the case $L_0 = \dfrac{p}{p-1}f$, which was treated in i) we conclude

$$\begin{aligned} \int_X \max(\mathcal{M}_\mathcal{T}\phi, L_0)^p d\mu &\leq F\omega_p\left(\frac{pL_0^{p-1}f - (p-1)L_0^p}{F}\right)^p \\ &= F[\omega_p(0)]^p = F\left(\frac{p}{p-1}\right)^p. \end{aligned}$$

The above imply that

$$I \leq L^p + F\left(\frac{p}{p-1}\right)^p - L_0^p = L^p + \left(\frac{p}{p-1}\right)^p (F - f^p),$$

which is our result in the second case. Lemma 4.2 is now proved. $\square$

5. Sharpness of Lemma 4.1

We suppose now that $L < \frac{p}{p-1} f$ and look at the relations (4.1) and (4.4). The first one is an inequality and states that

$$\int_X \max(\mathcal{M}_\mathcal{T}\phi, L)^p d\mu \le L^p - \frac{p}{p-1} L^{p-1} f + \frac{p}{p-1} \int_X \phi \max(\mathcal{M}_\mathcal{T}\phi, L)^{p-1} d\mu \tag{5.1}$$

while the second is an equality stating

$$\int_0^1 \max\left(\frac{1}{t}\int_0^t g, L\right)^p dt = L^p - \frac{p}{p-1} L^{p-1} f + \frac{p}{p-1} \int_0^1 g(t) \max\left(\frac{1}{t}\int_0^t g, L\right)^p dt. \tag{5.2}$$

We fix $g : (0,1] \to \mathbb{R}^+$. By Theorem 1 for

$$G_1(t) = \max(t, L)^p, \quad t \ge 0$$

$$G_2(t) = 1, \quad \text{and} \quad k = 1$$

we have that

$$\sup_{\phi^* = g} \int_X \max(\mathcal{M}_\mathcal{T}\phi, L)^p d\mu = v_g(L)$$

while for

$$G_1(t) = \max(t, L)^{p-1}, \quad t \ge 0$$

$$G_2(t) = t, \quad \text{and} \quad k = 1$$

we see that

$$\sup_{\phi^* = g} \int_X \phi \max(\mathcal{M}_\mathcal{T}\phi, L)^{p-1} d\mu = u_g(L).$$

That is if we leave the ϕ's to move along the rearrangements of g in (4.1) we produce the equality (4.4). During the proof of Lemma 4.1 we have also used the following inequality

$$\int_X \phi \max(\mathcal{M}_\mathcal{T}\phi, L)^{p-1} d\mu \le \left(\int_X \phi^p d\mu\right)^{1/p} \left(\int_X \max(\mathcal{M}_\mathcal{T}\phi, L)^p d\mu\right)^{p-1/p}. \tag{5.3}$$

For the proof of Lemma 4.2 we used inequalities only in the above two mentioned points. The first is attained if we use (4.1) and the discussion before. For the second we conclude that we need to find a sequence $g_n : (0,1] \to \mathbb{R}^+$ with $\int\limits_0^1 g_n(u) du = f$ and $\int\limits_0^1 g_n^p(u) du = F$ for which

$$\int_0^1 g_n(t) \max\left(\frac{1}{t}\int_0^t g_n, L\right)^{p-1} dt \approx \left(\int_0^1 g_n^p\right)^{1/p} \cdot \left(\int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L\right) dt\right)^{(p-1)/p}$$

that is we need equality in a Holder inequality. Therefore, we are forced to search for a $g : (0,1] \to \mathbb{R}^+$ with

$$\int_0^1 g(u) du = f \quad \text{and} \quad \int_0^1 g^p(u) du = F$$

for which

$$\max\left(\frac{1}{t}\int_0^t g, L\right) = cg(t), \quad \text{for} \quad t \in (0,1] \tag{5.4}$$

where

$$c = \omega_p\left(\frac{pL^{p-1}f - (p-1)L^p}{F}\right).$$

We state it as

Lemma 5.1. *There exists $g : (0,1] \to \mathbb{R}^+$ non-increasing, continuous for which the above three equations for the constants f, F and c hold, in case where $L < \dfrac{p}{p-1}f$.*

Proof. We set

$$g(t) = \begin{cases} Kt^{-1+\frac{1}{c}}, & \text{if} \quad t \in [0,\gamma] \\ \\ \frac{L}{c}, & \text{if} \quad t \in [\gamma, 1] \end{cases} \tag{5.5}$$

where γ and K are such that $\dfrac{1}{\gamma}\int\limits_0^\gamma g(u)du = L$, that is

$$Kc\gamma^{-1+\frac{1}{c}} = L. \tag{5.6}$$

It is obvious that g is continuous, non-increasing and satisfies (5.4). We are going to find now the constant γ in a way that

$$\int_0^1 g^p(u)du = F \Leftrightarrow \frac{K^p\left[t^{-p+\frac{p}{c}+1}\right]_{t=0}^{\gamma}}{\left(-p+\dfrac{p}{c}+1\right)} + \frac{L^p}{c^p}(1-\gamma) = F \Leftrightarrow$$

$$\frac{K^p c^p \gamma^{-p+\frac{p}{c}+1}}{c^p\left(-p+\dfrac{p}{c}+1\right)} + \frac{L^p}{c^p}(1-\gamma) = F$$

$$\Leftrightarrow \frac{c^p K^p \gamma^{-p+\frac{p}{c}+1}}{-(p-1)c^p + pc^{p-1}} + \frac{L^p}{c^p}(1-\gamma) = F. \tag{5.7}$$

Since (5.6) holds (5.7) becomes

$$\frac{L^p \cdot \gamma}{-(p-1)c^p + pc^{p-1}} + \frac{L^p}{c^p}(1-\gamma) = F. \tag{5.8}$$

By the definition of c we have that

$$-(p-1)c^p + pc^{p-1} = \frac{pL^{p-1}f - (p-1)L^p}{F} = b,$$

so (5.8) becomes

$$\frac{FL^p \cdot \gamma}{pL^{p-1}f - (p-1)L^p} + \frac{L^p}{c^p}(1-\gamma) = F \ \Leftrightarrow$$

$$\Leftrightarrow \ \gamma = \frac{F - L^p/c^p}{L^p\left(\dfrac{1}{b} - \dfrac{1}{c^p}\right)}.$$

We need to see that $\gamma \in [0,1]$. Obviously we have that

$$L^p \leq \int_X \max(\mathcal{M}_{\mathcal{T}}\phi, L)^p d\mu$$

for any ϕ such that $\int\limits_X \phi d\mu = f$ and $\int\limits_X \phi^p d\mu = F$. Additionally

$$\int_X \max(\mathcal{M}_{\mathcal{T}}\phi, L)^p d\mu \leq [\omega_p(b)]^p \cdot F = c^p F \Rightarrow F - L^p/c^p \geq 0.$$

Further c satisfies $-(p-1)c^p + pc^{p-1} = b$ as it is mentioned before thus $p(c^p - c^{p-1}) = c^p - b \Rightarrow c^p - b > 0 \Rightarrow \dfrac{1}{b} - \dfrac{1}{c^p} > 0$. From the above two inequalities we see that $\gamma \geq 0$ We prove now that $\gamma \leq 1 \Leftrightarrow$

$$F - \frac{L^p}{c^p} \leq \frac{L^p}{b} - \frac{L^p}{c^p} \Leftrightarrow$$

$$F \cdot b \leq L^p \Leftrightarrow F \cdot \frac{pL^{p-1}f - (p-1)L^p}{F} \leq L^p \Leftrightarrow L^{p-1}f \leq L^p,$$

which is true because of the fact that always $L \geq f$.

We consider now the function g as defined before with

$$\gamma = \frac{F - L^p/c^p}{L^p\Big(\dfrac{1}{b} - \dfrac{1}{c^p}\Big)} \in [0,1].$$

We prove that we additionally have that

$$\int_0^1 g(u)du = f \Leftrightarrow \int_0^\gamma Kt^{-1+\frac{1}{c}}dt + \frac{L}{c}(1-\gamma) = f$$

$$\Leftrightarrow Kc\gamma^{1/c} + \frac{L}{c}(1-\gamma) = f$$

$$\Leftrightarrow \ (\text{since } Kc = L\gamma^{1-\frac{1}{c}})$$

$$L\gamma + \frac{L}{c}(1-\gamma) = f \Leftrightarrow \gamma = \frac{f - L/c}{L\Big(1 - \dfrac{1}{c}\Big)},$$

So we need to check that

$$\frac{f - \dfrac{L}{c}}{L\Big(1 - \dfrac{1}{c}\Big)} = \frac{F - \dfrac{L^p}{c^p}}{L^p\Big(\dfrac{1}{b} - \dfrac{1}{c^p}\Big)} \Leftrightarrow$$

$$\frac{fc - L}{(c-1)} = \frac{Fc^p - L^p}{L^{p-1}\Big(\dfrac{c^p}{b} - 1\Big)} \Leftrightarrow$$

$$b = \frac{c^{p-1}(fc - L)L^{p-1}}{F(c^p - c^{p-1}) - L^p + fL^{p-1}}. \tag{5.9}$$

Because now of the relation

$$c^p - c^{p-1} = \frac{-b + c^p}{p},$$

(5.9) becomes

$$b = \frac{c^{p-1}(fc - L)L^{p-1}}{\frac{F}{p}(-b + c^p) - L^p + fL^{p-1}}. \tag{5.10}$$

On the other hand

$$\begin{aligned}\frac{F}{p}(-b + c^p) - L^p + fL^{p-1} &= \frac{F}{p}\bigg(-\frac{pL^{p-1}f - (p-1)L^p}{F} + c^p\bigg) - L^p + fL^{p-1}\\ &= -L^{p-1}f + \frac{p-1}{p}L^p + \frac{F}{p}c^p - L^p + fL^{p-1}\\ &= \frac{F}{p}c^p - \frac{L^p}{p} = \frac{Fc^p - L^p}{p}.\end{aligned}$$

Thus (5.10) is equivalent to

$$b = \frac{pc^{p-1}(fc - L)L^{p-1}}{Fc^p - L^p} \ \Leftrightarrow$$

$$\begin{aligned}&\Leftrightarrow \frac{pc^p f}{L} - pc^{p-1} = b\bigg(\frac{Fc^p}{L^p} - 1\bigg) \Leftrightarrow (\text{since } pc^{p-1} = b + (p-1)c^p)\\ &\Leftrightarrow \frac{pc^p f}{L} - b - (p-1)c^p = bF\frac{c^p}{L^p} - b \Leftrightarrow \frac{pf}{L} - (p-1) = b\frac{F}{L^p} \Leftrightarrow\end{aligned}$$

$$b = \frac{pL^{p-1}f - (p-1)L^p}{F}$$

which is true from the definition of b.

That is we derived Lemma 5.1. $\square$

We turn now to the case $L \geq \frac{p}{p-1}f$. For this one we need to construct a sequence $(g_n)_n$ with $g_n : (0,1] \to \mathbb{R}^+$ non-increasing and continuous such that

$$\int_0^1 g_n(u)du = f, \quad \int_0^1 g_n^p(u)du = F \quad \text{and}$$

$$\lim_n \int_0^1 \max\bigg(\frac{1}{t}\int_0^t g_n, L\bigg)^p dt \geq L^p + \bigg(\frac{p}{p-1}\bigg)^p (F - f^p)$$

where $L \geq \frac{p}{p-1}f$.

We set as before

$$g_n(t) = \begin{cases} k_n t^{-1+\frac{1}{c_n}}, & t \in (0, \gamma_n] \\ \dfrac{L_n}{c}, & t \in [\gamma_n, 1] \end{cases}$$

where $L_n \nearrow L_0 = \frac{p}{p-1}f$,

$$\gamma_n = \frac{F - L_n^p/c_n^p}{L_n^p\Big(\frac{1}{b_n} - \frac{1}{c_n^p}\Big)} = \frac{f - L_n/c_n}{L_n\Big(1 - \frac{1}{c_n}\Big)}$$

where $c_n = \omega_p(b_n)$, $b_n = \dfrac{pL_n^{p-1}f - (p-1)L_n^p}{F}$ and k_n is such that $k_n c_n \gamma_n^{-1+\frac{1}{c_n}} = L_n$.

Since $L_n \to L_0$ we have that $b_n \to 0$, $c_n \to \dfrac{p}{p-1}$ and $\gamma_n \searrow \dfrac{f - L_0\frac{p-1}{p}}{L_0\left(1-\frac{p}{p-1}\right)} = 0$.

According to the first case (where $L < \frac{p}{p-1}f$) we have that

$$\int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L_n\right)^p dt = [\omega_p(b_n)]^p F \ \to\ \left(\frac{p}{p-1}\right)^p F.$$

Now for $L \geq \frac{p}{p-1}f$,

$$\begin{aligned}
\int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L\right)^p dt &= L^p + \int_{\lambda=L}^{+\infty} p\lambda^{p-2}\left(\int_{\left\{u:\frac{1}{u}\int_0^u g_n>\lambda\right\}} g_n(u)du\right)d\lambda \\
&\overset{L\geq L_0}{=} L^p + \int_{\lambda=L_0}^{+\infty} p\lambda^{p-2}\left(\int_{\left\{u:\frac{1}{u}\int_0^t g_n>\lambda\right\}} g_n(u)du\right)d\lambda \\
&\quad - \int_{\lambda=L_0}^{L} p\lambda^{p-2}\left(\int_{\left\{u:\frac{1}{u}\int_0^u g_n>\lambda\right\}} g_n(u)du\right)d\lambda \\
&= L^p - L_0^p + \int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L_0\right)^p dt \\
&\quad - \int_{\lambda=L_0}^{L} p\lambda^{p-2}\left(\int_{\left\{u:\frac{1}{u}\int_0^u g_n>\lambda\right\}} g_n(u)du\right)d\lambda
\end{aligned} \tag{5.11}$$

By definition of the functions g_n we have that

$$\max\left(\frac{1}{t}\int_0^t g_n, L_n\right) = \omega_p(b_n)g_n(t).$$

Thus

$$\begin{aligned}
\int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L_0\right)^p dt &\geq \int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L_n\right)^p dt \\
&= [\omega_p(b_n)]^p \int_0^1 g_n^p(u)du = F[\omega_p(b_n)]^p, \quad \text{for every } n
\end{aligned}$$

and so

$$\lim_n \int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L_0\right)^p dt = F\left(\frac{p}{p-1}\right)^p.$$

At last

$$a_n(L) = \int_{\lambda=L_0}^{L} p\lambda^{p-2}\left(\int_{\left\{t:\frac{1}{t}\int_0^t g_n>\lambda\right\}} g_n(u)du\right)d\lambda$$

satisfies for a given $L \geq L_0$

$$\begin{aligned} a_n(L) &\leq \int_{\lambda=L_0}^{L} p\lambda^{p-2}\left(\int_{\left\{t:\frac{1}{t}\int\limits_0^t g_n>L_0\right\}} g_n(u)du\right)d\lambda \\ &= \left(\int_{\left\{t:\frac{1}{t}\int\limits_0^t g_n>L_0\right\}} g_n(u)du\right)\int_{\lambda=L_0}^{L} p\lambda^{p-2}d\lambda \\ &= \tau_L \cdot \int_{\left\{t:\frac{1}{t}\int\limits_0^t g_n>L_0\right\}} g_n(u)du. \end{aligned} \tag{5.12}$$

Note then that

$$\left|\left\{t \in (0,1] : \frac{1}{t}\int_0^t g_n \geq L_0\right\}\right| \leq \left|\left\{t \in (0,1] : \frac{1}{t}\int_0^t g_n \geq L_n\right\}\right| = \gamma_n,$$

because γ_n is the unique element of $(0,1]$ such that $\dfrac{1}{\gamma_n}\int\limits_0^{\gamma_n} g_n = L_n$.

Since $\gamma_n \to 0$, from (5.12) we deduce that $a_n(L) \to 0$, as $n \to \infty$, thus from (5.11)

$$\lim_n \int_0^1 \max\left(\frac{1}{t}\int_0^t g_n, L\right)^p dt \geq L^p - L_0^p + \left(\frac{p}{p-1}\right)^p F = L^p + \left(\frac{p}{p-1}\right)^p (F - f^p),$$

which is the result we needed to prove. From Lemma 5.1 and the calculations after it's proof we conclude the sharpness of Lemma 4.1.

6. Conclusions

By providing a generalization of the symmetrization principle given in [10] we give another proof of the computation for the Bellman function of three variables of the dyadic maximal operator,different from those given in [3] and [11].

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Further reading

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