Proceedings of the Workshop MAXBELLMAN 2760

Athens 2015

ΕΛΛΗΝΙΚΗ ΔΗΜΟΚΡΑΤΙΑ

Εθνικόν και Καποδιστριακόν Πανεπιστήμιον Αθηνών

ΕΙΔΙΚΟΣ ΛΟΓΑΡΙΑΣΜΟΣ ΚΟΝΔΥΛΙΩΝ ΕΡΕΥΝΑΣ
ΓΡΑΜΜΑΤΕΙΑ ΕΠΙΤΡΟΠΗΣ ΕΡΕΥΝΩΝ

Workshop MAXBELLMAN 2760
Thursday September 10, 2015
Room: G 31 Department of Mathematics
National and Kapodistrian University of Athens

The results announced in this workshop are from research that has been co-financed by the European Union and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF). ARISTEIA I, MAXBELLMAN 2760, research number 70/3/11913.

Programm of the Workshop

1) **10:15-11:00** Eleftherios Nikolidakis:
   Symmetrization principles for the dyadic maximal operator and stability results for its Bellman function
2) **11:15-12:00** Antonios Melas:
   Bellman functions related to Lorentz norms and lower norm estimates for the dyadic maximal operator
3) **12:15-13:00** Anastasios Delis:
   Sharp integral inequalities for the dyadic maximal operator and applications

   13:00-14:00: Lunch break

4) **14:15-15:00** Eleftherios Nikolidakis:
   Dyadic weights on homogeneous trees, reverse Holder inequalities and equimeasurable rearrangements of functions
5) **15:15-16:00** Antonios Melas:
   Stability theorems for the Bellman function of the dyadic maximal operator, an alternative approach

Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο

ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ
επένδυση στην κοινωνία της γνώσης
ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ ΚΑΙ ΘΡΗΣΚΕΥΜΑΤΩΝ
ΕΙΔΙΚΗ ΥΠΗΡΕΣΙΑ ΔΙΑΧΕΙΡΙΣΗΣ

ΕΣΠΑ 2007-2013
πρόγραμμα για την ανάπτυξη
ΕΥΡΩΠΑΪΚΟ ΚΟΙΝΩΝΙΚΟ ΤΑΜΕΙΟ

Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

Workshop Maxbellman 2760

Department of Mathematics, University of Athens, Room G21

September 10, 2015

Acknowledgement *This research has been co-financed by the European Union and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF). ARISTEIA I, MAXBELLMAN 2760, research number 70/3/11913.*

1 Eleftherios Nikolidakis: Symmetrization principles for the dyadic maximal operator and stability results for its Bellman function

It is well known that the dyadic maximal operator on $\mathbb{R}^n$ is a useful tool in analysis and is defined by

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(y)|\,dy : x \in Q,\ Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\right\}, \qquad (1)$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, for $N = 0, 1, 2, \ldots$. Its usefulness has first arised in [1] and [2].

It is also well known that it satisfies the following weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi > \lambda\}} |\phi(y)|\,dy, \qquad (2)$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$, and which is easily proved to be best possible. Certain refinements of (1.2) have been given in [16] and [17].

Using (1.2) it is not difficult to prove that the following L^p inequality is also true

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1}\|\phi\|_p, \qquad (3)$$

for every $p > 1$ and $\phi \in L^p(\mathbb{R}^n)$. Inequality (1.3) also turns to be best possible, as can be seen in [25].

Our wish is to refine (1.3). This was first done in [4] (and was continued in [5]), in the much more general setting of a non-atomic probability space (X, μ)

equipped with a tree structure $\mathcal{T}$, which is similar to the structure of the dyadic subcubes of $[0,1]^n$. The associated maximal operator is then defined by

$$M_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi|\, d\mu : x \in I \in \mathcal{T} \right\}, \tag{4}$$

for every $\phi \in L^1(X,\mu)$.

Then (1.2) and (1.3) still hold in this setting and remain sharp. In order to refine (1.3), for the general case of a tree $\mathcal{T}$, the so-called Bellman function of the dyadic maximal operator of two variables, given by

$$B_{\mathcal{T}}^{(p)}(f,F) = \sup\left\{ \int_X (M_{\mathcal{T}}\phi)^p \, d\mu : \phi \geq 0,\ \int_X \phi\, d\mu = f,\ \int_X \phi^p\, d\mu = F \right\}, \tag{5}$$

where $0 < f^p \leq F$, has to be evaluated. In [4] it is proved that

$$B_{\mathcal{T}}^{(p)}(f,F) = F\,\omega_p\left(\frac{f^p}{F}\right)^p,$$

where $\omega_p : [0,1] \to [1, \frac{p}{p-1}]$, is defined by $\omega_p(z) = H_p^{-1}(z)$, and $H_p(z)$ is given by $H_p(z) = -(p-1)z^p + pz^{p-1}$. As a consequence $B_{\mathcal{T}}^{(p)}(f,F)$ does not depend on the tree $\mathcal{T}$.

In [12] now it is shown that extremal functions do not exist for (1.5), in all non trivial cases. We are thus interested for the corresponding extremal sequences, concerning (1.5). More precisely we will say that $(\phi_n)_n$ is an extremal sequence for (1.5) if the following are satisfied: $\phi_n \geq 0$, $\int_X \phi_n\, d\mu = f$, $\int_X \phi_n^p\, d\mu = F$ for any $n \in N$, and additionally

$$lim \int_X (M_{\mathcal{T}}\phi_n)^p\, d\mu = F\,\omega_p\left(\frac{f^p}{F}\right)^p$$

as n tends to ∞.

The main core of [12] is the proof of the following

Theorem 1 *Let $(\phi_n)_n$ be an extremal sequence as above. Then for every $I \in \mathcal{T}$, the following are satisfied: $lim \frac{1}{\mu(I)} \int_I \phi_n\, d\mu = f$ and $lim \frac{1}{\mu(I)} \int_I \phi_n^p\, d\mu = F$, as n tends to ∞. Additionally the identity*

$$lim \frac{1}{\mu(I)} \int_I (M_{\mathcal{T}}\phi_n)^p\, d\mu = F\,\omega_p\left(\frac{f^p}{F}\right)^p,$$

holds for any such I.

As an application of the above Theorem it is not difficult to prove the weak-L^p uniqueness of any such extremal sequence. This also can be seen in [12].

In [13] now we consider $g, h : (0,1] \to R^+$, non increasing functions and $G : R^+ \to R^+$ non decreasing. Then the following holds and is presented in the above mentioned article.

Theorem 2 *For any $k \in (0,1]$, the following formula is true:*

$$\sup\left\{\int_K G[(M_{\mathcal{T}}\phi^*)]h(t)\,dt : \phi^* = g,\ K \text{ measurable subset of } (0,1],\ |K| = k\right\}$$

$$= \int_0^k G(\frac{1}{t}\int_0^t g)h(t)\,dt.$$

In [13] we also present some non-trivial applications of the above symmetrization principle. More precisely we prove the following.

Theorem 3 *The following inequality is true and sharp for every q, p such that $q < p$, $p > 1$:*

$$||M_{\mathcal{T}}\phi||_{L^{p,q}} \leq \frac{p}{p-1}||\phi||_{L^{p,q}},$$

where $||.||_{L^{p,q}}$ stands for the standard Lorentz quasinorm on $L^{p,q}$.

We continue the description of this talk by stating the following generalized symmetrization principle, which is presented in [18].

Theorem 4 *For any $k \in (0,1]$, the following equality is true:*

$$\sup\left\{\int_K G_1(M_{\mathcal{T}}\phi)G_2(\phi)\,d\mu : \phi^* = g,\ K \text{ measurable subset of } X,\ \mu(K) = k\right\}$$

$$= \int_0^k G_1(\frac{1}{t}\int_0^t g)G_2(g(t))\,dt.$$

where $G_i : [0,+\infty] \to [0,+\infty]$ are increasing functions for $i = 1,2$, while $g : (0,1] \to R^+$ is non increasing.

As an application we prove in [18] the following.

Theorem 5 *For every f and F such that $0 < f^p \leq F$ and $L \geq f$, we have that*

$$\int_X max(M_{\mathcal{T}}\phi, L)\,d\mu \leq \begin{cases} F\omega_p(\dfrac{pL^{p-1}f - (p-1)L^p}{F})^p, & \text{if} \quad L < \dfrac{p}{p-1}f \\ L^p + (\dfrac{p}{p-1})^p(F - f^p), & \text{if} \quad L > \dfrac{p}{p-1}f. \end{cases}$$

for every ϕ such that $\int_X \phi\,d\mu = f$ and $\int_X \phi^p\,d\mu = F$. Moreover the inequality (1.6) is best possible for both ranges of L.

Finally in our work (see [15]), connected with stability results related to the Bellman function of the dyadic maximal operator (1.5), we prove the following.

Theorem 6 *Let $(\phi_n)_n$ be a sequence of non negative, $\mathcal{T}-$ good functions (as defined in [4]), such that $\int_X \phi_n \, d\mu = f$ and $\int_X \phi_n^p \, d\mu = F$. Then $(\phi_n)_n$ is an extremal sequence for (1.5), if and only if*

$$lim \int_X |M_{\mathcal{T}}\phi_n - c\phi_n|^p \, d\mu = 0,$$

as n tends to ∞, where $c = \omega_p\left(\frac{f^p}{F}\right)$.

That is $(\phi_n)_n$ is an extremal sequence for (1.5), if and only if its terms behave approximately, in L^p, like eigenfunctions of $M_{\mathcal{T}}$, for the eigenvalue $c = \omega_p\left(\frac{f^p}{F}\right)$.

2 Antonios Melas: Bellman Functions Related to Lorentz norms and lower estimates for the dyadic maximal operator

The dyadic maximal operator on $\mathbb{R}^n$

$$\begin{aligned} M_{\,d}\phi(x) &= \sup\{\frac{1}{|Q|}\int_Q |\phi(u)| \, du : \\ x &\in Q, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\} \end{aligned}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ where the dyadic cubes are the cubes formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, 2, ...$ satisfies

$$|\{x \in \mathbb{R}^n : M_{\,d}\phi(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{M_{\,d}\phi > \lambda\}} |\phi(u)| \, du.$$

$$\|M_d\phi\|_p \leq \frac{p}{p-1}\|\phi\|_p$$

for every $p > 1$ and every $\phi \in L^p(\mathbb{R}^n)$. This is a particular case of Doob's inequality for martingales, and has been shown to be sharp by Burkholder and Wang.

The study of deeper properties of such operators though comes from the introduction of the so called Bellman functions, done by Nazarov Treil Volberg.

Such a function is defined for any $p > 1$ by

$$\begin{gathered} \mathcal{B}_p(F, f, L) = \sup\{\frac{1}{|Q|}\int_Q (M_d\phi)^p : \\ \text{Av}_Q(\phi^p) = F, \text{Av}_Q(\phi) = f, \\ \sup_{R:Q\subseteq R} \text{Av}_R(\phi) = L\} \end{gathered}$$

where Q is a fixed dyadic cube, R runs over all dyadic cubes containing Q, ϕ is nonnegative in $L^p(Q)$ and the variables F, f, L satisfy $0 \leq f \leq L, f^p \leq F$. $\mathcal{B}_p$

is independent of the choice of Q (so we may take $Q = [0,1]^n$) and satisfies a certain "pseudoconvexity" inequality.

The context of dyadic cubes can be generalizd in the following:

Let (X, μ) be a nonatomic probability space (i.e. $\mu(X) = 1$). Two measurable subsets A, B of X will be called almost disjoint if $\mu(A \cap B) = 0$. Then we give the following.

Definition 7 *A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:*

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there corresponds a finite subset $\mathcal{C}(I) \subseteq \mathcal{T}$ containing at least two elements such that:

(a) the elements of $\mathcal{C}(I)$ are pairwise almost disjoint subsets of I,

(b) $I = \bigcup \mathcal{C}(I)$.

(iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} \mathcal{C}(I)$.

(iv) We have $\lim\limits_{m \to \infty} \sup\limits_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$.

Examples 1) If Q_0 is the unit cube $\mathbb{R}^n$ then the set of all dyadic cubes $Q \subseteq Q_0$ is a tree with $\mathcal{C}(Q)$ being the set of the 2^n subcubes of Q obtained by bisecting its sides. More generally for any integer $m > 1$ the set of all m-adic cubes $Q \subseteq Q_0$ is a tree with $\mathcal{C}(Q)$ being the set of the m^n subcubes of Q obtained by dividing each side of it into m equal parts.

2) Given the integers $d_1, ..., d_n \geq 1$ and $m > 1$ we can define a tree $\mathcal{T}$ of parallilepipeds on $X = Q_0$ by setting for each parallilepiped R the family $\mathcal{C}(R)$ to consist of the parallilepipeds formed by dividing the dimensions of R into $m^{d_1}, ..., m^{d_n}$ equal parts respectively. This tree is related to nonisotropic dilations. For examle if $n = 2, m = 2, d_1 = 1$ and $d_2 = 2$ we get the set of dyadic parabolic rectangles contained in $[0,1]^2$.

3) On $X = [0,1]$ let $\beta > 0$ and for each $I \subseteq X$ we let $\mathcal{C}(I)$ consist by the two subintervals of I formed by dividing it in ratio β. Then using the relation in (iii) in the above definition we get a tree on X. For $\beta = 1$ we get the dyadic intervals. Actually it is not hard to see that any tree in a general space X can in a sense be modeled in the space $[0,1]$ with the Lebesgue measure, but we will not use that.

Then we can define the maximal operator associated to $\mathcal{T}$ as follows

$$M_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in \mathcal{T} \right\}$$

for every $\phi \in L^1(X, \mu)$.

The above maximal operator satisfies essentially the same inequalities as M_d, the proof being a trivial adaptation of the proof in the dyadic case. Now we define the corresponding Bellman function as

$$\mathcal{B}_p^{\mathcal{T}}(F,f,L) = \sup\{\int_X (\max(M_{\mathcal{T}}\phi, L))^p d\mu :$$
$$\phi \geq 0, \phi \in L^p(X,\mu),$$
$$\int_X \phi^p d\mu = F, \int_X \phi d\mu = f\}.$$

Define for any $p > 1$ the function

$$H_p(z) = -(p-1)z^p + pz^{p-1}$$

defined for $z \in [1, \frac{p}{p-1}]$. It is easy to see that H_p is strictly decreasing on this interval and it maps it onto $[0,1]$. We now let $\omega_p : [0,1] \to [1, \frac{p}{p-1}]$ denote the inverse function H_p^{-1} of H_p. Then we have the following.

Theorem 8 *[4] For any nonatomic probability space (X,μ), any tree-like family $\mathcal{T}$ and any $p > 1$ the corresponding Bellman function is given by*

$$\mathcal{B}_p^{\mathcal{T}}(F,f,L) =$$
$$\{ \begin{array}{l} F\omega_p\left(\dfrac{pL^{p-1}f-(p-1)L^p}{F}\right)^p \ \ if\ L < \dfrac{p}{p-1}f \\ L^p + (\dfrac{p}{p-1})^p(F-f^p)\ \ if\ L \geq \dfrac{p}{p-1}f. \end{array}$$

For example

$$\mathcal{B}_2^{\mathcal{T}}(F,f,L) =$$
$$\{ \begin{array}{l} (\sqrt{F} + \sqrt{F + L^2 - 2Lf})^2 \text{ if } L < 2f \\ L^2 + 4(F - f^2) \text{ if } L \geq 2f. \end{array}$$

so

$$\|M_{\mathcal{T}}\phi\|_2 \leq \|\phi\|_2 + (\|\phi\|_2^2 - \|\phi\|_1^2)^{1/2}$$

and this is sharp stronger form of Doobs inequality for $p = 2$.

The proof is based on a combinatorial analysis of the operator and an effective **linearization for step functions**.

2.1 Lorentz Norms

However in order to treat more general norms than the L^p for example Lorentz norms, new ideas are needed. A method to treat such problems as well as mixed norm estimates was the introduction of the Hardy operator on decreasing function by A. Melas in 2009 through a process of symmetrization. This that can be called **symmetrization principle** has been later refined by E. Nikolidakis who showed that it can be carried throught the symmetric equimeasurable function of the test function. This gave the following general.

Theorem 9 *[7] Let $G : [0, +\infty) \to [0, +\infty)$ be non-decrasing, $h : (0, 1] \to \mathbb{R}^+$ be any locally integrable function. Then for any nonatomic probability space (X, μ), equipped with any tree-like family $\mathcal{T}$, for any non-increasing right continuous integrable function $g : (0, 1] \to \mathbb{R}^+$ and any $k \in (0, 1]$, the following equality holds:*

$$\sup\{\int_0^k G[(M_{\mathcal{T}}\phi)^*(t)]h(t)dt : \\ \phi \textit{ measurable on } X \textit{ with } \phi^* = g\} = \\ = \int_0^k G\left(\frac{1}{t}\int_0^t g(u)du\right) h(t)dt.$$

Here ϕ^* denotes the equimeasurable decreasing rearrangement of the measurable function $\phi : X \to \mathbb{R}$ which is defined on $(0, 1]$ since X is a probability space.

This then gives the following.

Theorem 10 *[7] Given the real numbers $f, F_1, ..., F_m > 0$ and $p_1, ..., p_m > 1$ with $f \leq \min\{\left(\frac{p_j}{p_j-1}F_j\right)^{1/p_j} : 1 \leq j \leq m\}$ and given any nondecreasing $G : [0, +\infty) \to [0, +\infty)$ and $h : (0, 1] \to \mathbb{R}^+$ be any locally integrable function, we have for the following Bellman type function*

$$\mathcal{B}^{\mathcal{T}}_{G,h,p_1,...,p_m}(F_1, ...F_m, f, k) = \\ \sup\{\int_0^k G[(M_{\mathcal{T}}\phi)^*(t)]h(t)dt : \\ \phi \geq 0 \textit{ measurable on } X \\ \textit{with } \|\phi\|_1 = f, \ \|\phi\|_{p_1,\infty} \leq F_1, ..., \|\phi\|_{p_m,\infty} \leq F_m\}$$

the equality

$$\mathcal{B}^{\mathcal{T}}_{G,h,p_1,...,p_k}(F_1, ...F_k, f, k) = \\ = \int_0^k G\left(\frac{1}{t}\int_0^{\min(t,\sigma)} \min_{1\leq j\leq m}\left(\frac{F_j}{u}\right)^{1/p_j} du\right) h(t)dt$$

where σ is defined by the equality

$$\int_0^\sigma \min_{1\leq j\leq m}\left(\frac{F_j}{u}\right)^{1/p_j} du = f.$$

Using the above Theorem we have found the $L^{p,\infty} \to L^{q,r}$ Lorentz type Bellman function for the maximal operator where we denote by p', q' the dual exponents of $p, q > 1$ (so $p' = \frac{p}{p-1}$)

Theorem 11 *[7] Given $1 < q < p$ and $r > 0$ the Bellman function:*

$$\mathcal{B}^{\mathcal{T}}_{(p,\infty),(q,r)}(F,f,L) = \sup\{\|\max(M_{\mathcal{T}}\phi, L)\|^r_{L^{q,r}(X,\mu)} : \\ \phi \geq 0 \textit{ is measurable with} \\ \|\phi\|_{L^1(X,\mu)} = f, \ \|\phi\|^p_{L^{p,\infty}(X\mu)} = F\}$$

defined for $0 < f < p'F^{1/p}$ and $f \leq L$ is given by

$$\mathcal{B}^{\mathcal{T}}_{(p,\infty),(q,r)}(F,f,L) = \\ \begin{cases} \dfrac{q(p-1)q'}{r(p-q)}(p')^{p'r/q'} f^{\frac{r(p-q)}{q(p-1)}} F^{\frac{r(q-1)}{q(p-1)}} + \\ +\dfrac{q}{r}L^r - \dfrac{q}{r}q' f^{\frac{r}{q}} L^{\frac{r}{q}(q-1)} \\ \textit{when } L \leq (p')^{p'} \left(\frac{F}{f}\right)^{1/(p-1)} \\ \dfrac{q(p')^{rp/q}}{r(\frac{p}{q}-1)} F^{\frac{r}{q}} L^{r(1-\frac{p}{q})} + L^r \\ \textit{when } L \geq (p')^{p'} \left(\frac{F}{f}\right)^{1/(p-1)} \end{cases}.$$

Next we define the Bellman function related to a Lorentz $L^{p,q} \to L^{p,q}$ type estimate for the (martingale) maximal operator, where $p, \dot{q} > 1$ are arbitrary

$$\mathcal{BL}^{\mathcal{T}}_{p,q}(F,f) = \sup\{\|M_{\mathcal{T}}\phi\|^q_{L^{p,q}(X,\mu)} : \\ \phi \geq 0 \text{ is measurable with} \\ \|\phi\|_{L^1(X,\mu)} = f, \ \|\phi\|^q_{L^{p,q}(X\mu)} = F\}.$$

Theorem 12 *[7] The above Bellman function is defined for all pairs (F,f) with (i) $0 < f^q \leq \left(\frac{p'}{q'}\right)^{q-1} F$ if $1 < p \leq q$ and (ii) $0 < f^q \leq \frac{q}{p}F$ if $1 < q < p$ and in both cases it is given by*

$$\mathcal{BL}^{\mathcal{T}}_{p,q}(F,f) = \left(\frac{p'}{q'}\right)^q \omega_q \left(\left(\frac{q'}{p'}\right)^{q-1} \frac{f^q}{F}\right)^q F.$$

Here $\omega_q : [0,1] \to [1,q']$ is the inverse function of $H_q(z) = -(q-1)z^q + qz^{q-1}$ (defined on $[1,q']$) thus the same function as the one appearing in the Bellman functions of the usual L^p norms.

2.2 Infimum

Now we turn to sharp forms of the lower estimates and define the following function

$$\mathcal{B}^{\mathcal{T}}(F,f,L)=\inf\{\int_X \max(M_{\mathcal{T}}\phi,L)^p d\mu :$$
$$\phi\geq 0 \text{ measurable with}$$
$$\int_X \phi d\mu = f,\quad \int_X \phi^p d\mu = F\}.$$

Then our first main theorem is the following.

Theorem 13 *[9] For any nonatomic probability space (X,μ), any N-homogeneous tree-like family $\mathcal{T}$ and any F,f,L with $f\leq L$ and $f^p\leq F$ the corresponding Bellman function is given by*

$$\mathcal{B}^{\mathcal{T}}(F,f,L)=L^p+\frac{N^p-1}{N^p-N}(F-L^{p-1}f)^+$$

where $x^+=\max(x,0)$.

To get bounds for weak norms we first define the following Bellman type function

$$\mathcal{D}_p^{\mathcal{T}}(F,f,\kappa)=\inf\{\sup_{\mu(E)=\kappa}\int_E (M_{\mathcal{T}}\phi)^p d\mu :$$
$$\phi\geq 0 \text{ measurable with}$$
$$\int_X \phi d\mu = f,\quad \int_X \phi^p d\mu = F\}.$$

the inner supremum taken over all measurable subsets E of X having measure κ, where $\kappa\in(0,1]$, and the positive numbers F,f are such that $f^p\leq F$. Then we will prove the following

Theorem 14 *[8] For any N-homogeneous tree-like family $\mathcal{T}$ any $p>1$ any F,f with $f^p\leq F$ and any $\kappa\in(0,1]$ we have*

$$\begin{aligned}\mathcal{D}_p^{\mathcal{T}}(F,f,\kappa) &= \min\{\kappa u^p+\frac{N^p-1}{N^p-N}(F-u^{p-1}f):\\ f &\leq u\leq \min((\frac{F}{f})^{1/(p-1)},\frac{f}{\kappa})\}\end{aligned}$$

and writing $c(N,p)=\frac{p-1}{p}\frac{N^p-1}{N^p-N}<1$,

$$\mathcal{D}_p^{\mathcal{T}}(F,f,\kappa)=$$
$$=\begin{cases}\kappa f^p+\frac{N^p-1}{N^p-N}(F-f^p)\\ \text{if } \ c(N,p)\leq\kappa\leq 1\\ \frac{N^p-1}{N^p-N}(F-c(N,p)^{p-1}\frac{f^p}{p\kappa^{p-1}})\\ \text{if } \ c(N,p)(\frac{f^p}{F})^{1/(p-1)}\leq\kappa\leq c(N,p)\\ \kappa(\frac{F}{f})^{p/(p-1)}\\ \text{if } \ 0<\kappa\leq c(N,p)(\frac{f^p}{F})^{1/(p-1)}.\end{cases}$$

From the above theorem one obtains lower bounds for the following equivalent norm on weak L^q when $q > p$:

$$\|\psi\|_{q,\infty} = \sup_{0<\mu(E)} \mu(E)^{-\frac{1}{p}+\frac{1}{q}} \left(\int_E |\psi|^p \, d\mu\right)^{1/p}$$

Theorem 15 *[8] Given $q > p > 1$ and $F, f > 0$ with $f^p \leq F$ we have for any measurable $\phi \geq 0$ on X with $\int_X \phi d\mu = f,\ \int_X \phi^p d\mu = F$ the following: i) If $\frac{q-1}{q-p}\frac{f^p}{F} < 1$*

$$\begin{gathered}\|M_{\mathcal{T}}\phi\|_{q,\infty} \geq \\ \max[c(N,p)^{1/q}\frac{(q-p)^{(q-p)/q(p-1)}}{(q-1)^{(q-1)/q(p-1)}}(\frac{F^{\frac{q-1}{p-1}}}{f^{\frac{q-p}{p-1}}})^{1/q}, \\ (f^p + \frac{N^p-1}{N^p-N}(F-f^p))^{1/p}].\end{gathered}$$

ii) If $\frac{q-1}{q-p}\frac{f^p}{F} \geq 1$

$$\begin{gathered}\|M_{\mathcal{T}}\phi\|_{q,\infty} \geq \\ \max[c(N,p)^{1/q}(\frac{p}{p-1})^{1/p}(F-\frac{f^p}{p})^{1/p}, \\ (f^p + \frac{N^p-1}{N^p-N}(F-f^p))^{1/p}].\end{gathered}$$

Next we examine the strong mixed-norms lower bounds considering the following Bellman type function

$$\begin{aligned}\mathcal{B}^{\mathcal{T}}_{p,q}(F,f) &= \inf\{\int_X (M_{\mathcal{T}}\phi)^q d\mu : \\ \phi &\geq 0 \text{ measurable}, \\ \int_X \phi d\mu &= f,\ \int_X \phi^p d\mu = F\}.\end{aligned}$$

Proposition 16 *For any N-homogeneous tree-like family $\mathcal{T}$ any $p > 1$ any $q < p$ and any F, f with $f^p \leq F$ we have*

$$\mathcal{B}^{\mathcal{T}}_{p,q}(F,f) = f^q.$$

Thus the interesting case is when $q > p$ and in this case we will prove the following

Theorem 17 *[8] For any N-homogeneous tree-like family $\mathcal{T}$ any $p > 1$ any $q > p$ and any F, f with $f^p \leq F$ we have*

$$\mathcal{B}^{\mathcal{T}}_{p,q}(F,f) \geq f^q + \frac{N^q-1}{N^q-N}(\frac{F^{\frac{q-1}{p-1}}}{f^{\frac{q-p}{p-1}}} - f^q)$$

and we have equality when $(F/f^p)^{1/(p-1)}$ is a power of N, that is if m is a nonnegative integer then

$$\mathcal{B}_{p,q}^{\mathcal{T}}(N^{m(p-1)}f^p, f) = f^q[1 + \frac{N^q - 1}{N^q - N}(N^{m(q-1)} - 1)].$$

3 Anastasios Delis: Sharp integral inequalities for the dyadic maximal operator and applications

As it was mentioned in the first talk by Eleftherios Nikolidakis we are interested for the evaluation of (1.5). In [18] now, a different approach has been given, for this purpose. In the process of that proof, the following intermediate inequality regarding the generalized maximal operator occurred.

Theorem 18 *Let $\phi \in L^p(X,\mu)$ be non-negative, with $\int_X \phi\, d\mu = f$. Then the following inequality is true*

$$\int_X (M_{\mathcal{T}}\phi)^p\, d\mu \leq -\frac{1}{p-1} f^p + \frac{p}{p-1}\int_X \phi\, (M_{\mathcal{T}}\phi)^{p-1}\, d\mu. \tag{6}$$

Our intension is to refine (1.5) even further, that is to insert the L^q-norm of ϕ as an independent variable in (1.5), and try to find the best possible upper bound of $\|\mathcal{M}_d\phi\|_p$, when the L^1, L^q and L^p norms of ϕ are given.

The goal of this lecture will be to describe briefly the proofs of two inequalities satisfied by the generalized maximal operator that we hope to provide the intermediate step in proving the aforementioned upper bound, in the same way that (3.1) did for (1.5). These results are described in [3].

The first inequality states that for ω_q, which is defined as above, with q in place of p, the generalization of the above Theorem holds.

Theorem 19 *Let $q \in (1,p]$, $f > 0$, $A > 0$ with $f^q < A$ and $\phi \in L^p(X,\mu)$ non-negative, with $\int_X \phi\, d\mu = f$ and $\int_X \phi^q\, d\mu = A$. Then the following inequality holds*

$$\int_X (M_{\mathcal{T}}\phi)^p\, d\mu \leq -\frac{q(\beta+1)}{G(p,q,\beta)} f^p + \tag{7}$$
$$+\frac{p(\beta+1)^q}{G(p,q,\beta)}\left(\int_X (M_{\mathcal{T}}\phi)^p\, d\mu\right)^{\frac{p-q}{p}}\left(\int_X \phi^p\, d\mu\right)^{\frac{q}{p}},$$

for every $\beta > 0$, where $G(p,q,\beta) = p(q-1)\beta + (p-q)(\beta+1)$. Additionally (3.2) is best possible for $\beta = \omega_q(\frac{f^q}{A}) - 1$ for any fixed choices of f, A.

The second one relates the p and q-norms, $(1 < q < p,)$ of the maximal function with the 1 and q, norms of this function. Specifically, for $G(p,q,\beta)$ as above, we have the following.

Theorem 20 *Let $q \in [1,p]$, $f > 0$, $A > 0$ with $f^q < A$ and $\phi \in L^p(X,\mu)$ non-negative, with $\int_X \phi \, \mathrm{d}\mu = f$ and $\int_X \phi^q \, \mathrm{d}\mu = A$. Then the following inequality holds*

$$\int_X (M_{\mathcal{T}}\phi)^p \, d\mu \le \frac{p(\beta+1)^q}{G(p,q,\beta)} \int_X (M_{\mathcal{T}}\phi)^{p-q} \phi^q \, d\mu + \frac{(p-q)(\beta+1)}{G(p,q,\beta)} f^p + \frac{p(q-1)\beta}{G(p,q,\beta)} f^{p-q} \int_X (M_{\mathcal{T}}\phi)^q \, d\mu - \frac{p(\beta+1)^q}{G(p,q,\beta)} f^{p-q} A, \tag{8}$$

for every $\beta > 0$. Additionally (3.3) is best possible for $\beta = \omega_q(\frac{f^q}{A}) - 1$.

In [3] now certain applications are given of the above mentioned inequalities. Concerning (3.2) we prove that this generalized inequality is responsible for the exact evaluation of the Bellman function (1.5). In fact if the 1 and p, norms of a function ϕ are given we prove in [3], that for a certain value of β, which depends only on these norms and the value of q we can apply the technique that arises in [18], and produce in the same manner the desired value of (1.5). That is the generalized inequality (3.2) can also be used for the determination of (1.5) for any value of q.

As an application of (3.3), we provide an inequality, with the parameters involved being the same as in (3.2), but which has as a consequence the validity of the last mentioned inequality, by a simple application of Holder's inequality, and which also generalizes (3.1) when we let q tend to 1.

We mention also that certain Bellman functions corresponding to several problems in harmonic analysis, have been studied in [10], [11], [21], [22], [23] and [24]. Also a different approach for the study of Bellman functions and related inequalities can be seen in [20].

4 Eleftherios Nikolidakis: Dyadic weights on homogeneous trees, reverse Holder inequalities and equimeasurable rearrangements of functions

It is a known fact that if ϕ is a non negative Lesbesgue integrable function defined on $(0,1]$, which satisfies

$$\frac{1}{|I|} \int_I \phi(y) \, dy \le c.essinf_I(\phi), \tag{9}$$

for any I subinterval of $[0,1]$, then its non increasing rearrangement satisfies the same inequality (with the same constant c), or equivalently the inequality

$$\frac{1}{t} \int_0^t \phi^*(y) \, dy \le c\phi^*(t), \tag{10}$$

is true for every $t \in (0,1]$. This last equivalence is true because of the type of monotonicity of ϕ^*.

When inequality (2.1) holds, we say that ϕ is an A_1 weight on $(0,1]$, with A_1 constant not more than c, and we write $[\phi]_{A_1} \leq c$. If (2.1) is sharp, in the sense that the supremum, of its left side divided by $essinf_I(\phi)$, for all subintervals I of $[0,1]$, is equal to c, then we say that the A_1 constant of ϕ equals c and we write $[\phi]_{A_1} = c$. Thus the above statement indicates that if $[\phi]_{A_1} \leq c$ then $[\phi^*]_{A_1} \leq c$.

We consider now a non negative integrable function defined on $[0,1]^n$, for some $n \geq 1$, which satisfies the following inequality

$$\frac{1}{|D|}\int_D \phi(y)\,dy \leq c.essinf_D(\phi), \tag{11}$$

for every $D \in \mathcal{T}_{2^n}$, for some fixed constant $c \geq 1$. Here $\mathcal{T}_{2^n}$ denotes the usual homogeneous tree of all dyadic subcubes of $[0,1]^n$. We say then that ϕ is a dyadic A_1 weight, with dyadic A_1-constant less or equal to c and write $[\phi]^d_{A_1} \leq c$. Similar definition as before is given for the those dyadic weights with dyadic A_1-constant equal to c.

We wish now to know whether there exists an absolute constant K, depending only on c and n, such that if we pass from ϕ to ϕ^*, which is now defined on $(0,1]$, then this function satisfies the analogous to (2.2) inequality:

$$\frac{1}{t}\int_0^t \phi^*(y)\,dy \leq K.\phi^*(t), \tag{12}$$

that is ϕ^* is a usual A_1 weight on $(0,1]$, with A_1-constant less or equal to K.

In this direction we prove the following result which is presented in [14] in more generality, that is for functions defined on the whole R^n.

Theorem 21 *Let ϕ be a dyadic A_1 weight on $[0,1]^n$, such that $[\phi]^d_{A_1} = c$. Then the inequality*

$$\frac{1}{t}\int_0^t \phi^*(y)\,dy \leq (2^n c - 2^n + 1)\phi^*(t), \tag{13}$$

is satisfied, for every $t \in (0,1]$. Moreover the last inequality is sharp when we consider all dyadic A_1-weights with $[\phi]^d_{A_1} = c$

That is we prove that the best possible constant K, which we are searching for, is finite and equals $K(c,n) = 2^n c - 2^n + 1$.

In our last work (see [19]), we devote our study to those weights $\phi : Q_0 = [0,1]^n \to R^+$, that satisfy the inequality

$$\frac{1}{|Q|}\int_Q \phi^p(y)\,dy \leq (\frac{1}{|Q|}\int_Q \phi(y)\,dy)^p, \tag{14}$$

for every $Q \in \mathcal{T}_{2^n}$, an exponent $p > 1$, and some fixed constant $c \geq 1$, which is independent of Q. In the following Theorem we prove that ϕ^* satisfies a reverse Holder inequality with exponent p, and constant $2^n c - 2^n + 1$, on all subintervals of $[0,1]$ of the form $(0,t]$. We state is as follows.

Theorem 22 *Let ϕ be a dyadic A_1 weight on $[0,1]^n$, such that (2.6) is true for some $p, c > 1$, and every cube $Q \in \mathcal{T}_{2^n}$. Then the non increasing rearrangement of ϕ, denoted by ϕ^*, satisfies the following inequality*

$$\frac{1}{t}\int_0^t (\phi^*(y))^p\, dy \leq (2^n c - 2^n + 1)(\frac{1}{t}\int_0^t (\phi^*(y))\, dy)^p, \tag{15}$$

for every $t \in (0,1]$.

As a consequence, due to well known theory of weights defined on R, we conclude that there exists an absolute constant $e = e(n,p,c) > 0$, such that ϕ^*, belongs to $L^q([0,1])$, for any $q \in [p, p+e)$, and so ϕ belongs to $L^q([0,1]^n)$ for any q in this range. That is we find an explicit value of e, which enables us to increase the integrability properties of ϕ.

5 Antonios Melas: Stability theorems of the dyadic maximal operator, an alternative approach

In analyzing this estimate more deeply one is lead to consider what properties have the extremals or approximate extremals for it. In this direction it has been proved by Nikolidakis (for fixed F, f, p) that if a sequence (ϕ_n) of nonnegative functions is extremal in the sense that

$$\int_X \phi_n^p d\mu = F, \int_X \phi_n d\mu = f$$
$$\text{and } \lim_n \int_X (M_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p\left(\frac{f^p}{F}\right)^p$$

then in the limit the sequence behaves like an approximate "eigenfunction" of $M_{\mathcal{T}}$ meaning that

$$\lim_n \int_X |M_{\mathcal{T}}\phi_n - c\phi_n|^p\, d\mu = 0$$

where $c = \omega_p\left(\frac{f^p}{F}\right)$.

Theorem 23 *[6] Let $p \geq 2$ be given. Then there exists an absolute constant C_p such that: If $(X, \mu, \mathcal{T})$ is a nonatomic probability space equipped with a tree-like family, if $F, f > 0$ are real numbers with $f < F^{1/p}$ and if $\delta > 0$ is sufficiently small then for any nonnegative function $\phi \in L^p(X)$ satisfying $\int_X \phi^p d\mu = F, \int_X \phi d\mu = f$ and $\int_X (M_{\mathcal{T}}\phi)^p d\mu \geq (1-\delta)F\omega_p\left(\frac{f^p}{F}\right)^p$ the following holds*

$$\int_X |M_{\mathcal{T}}\phi - c\phi|^p\, d\mu \leq C_p F\delta$$

where $c = \omega_p\left(\frac{f^p}{F}\right)$.

Corollary 24 *[6] Let $p \geq 2$ be given. Then there exists a absolute constants A_p and B_p such that: If $(X, \mu, \mathcal{T})$ is a nonatomic probability space equipped with a tree-like family and if $\phi \in L^p(X)$ is a nonnegative function satisfying*

$\|M_d \phi\|_p \geq (\frac{p}{p-1} - \varepsilon) \|\phi\|_p$ where $\varepsilon > 0$ is sufficiently small then we have:

$$\|\phi\|_1^p \leq A_p \|\phi\|_p^p \varepsilon \text{ and } \left\| M_{\mathcal{T}} \phi - \frac{p}{p-1} \phi \right\|_p^p \leq B_p \|\phi\|_p^p \varepsilon.$$

The proof of Theorem 1 uses the combinatorial approach for the Bellman function $\mathcal{B}_p^{\mathcal{T}}$ combined with the following inequalities similar to the well known Clarkson's inequalities which is the reason we have this result for the range $p \geq 2$..

Lemma 25 *Let $p \geq 2$ be given. Then*

(i) For all $s, t \geq 0$ we have

$$t^p - s^p \geq |t - s|^p + p(t - s)s^{p-1}.$$

(ii) If (X, μ) is a nonatomic probability space and if $h \in L^p(X)$ is nonnegative then

$$\int_X h^p d\mu - (\int_X h d\mu)^p \geq \int_X \left| h - \int_X h d\mu \right|^p d\mu.$$

(iii) If $x, y, \lambda, \mu > 0$ then

$$\lambda x^p + \mu y^p - (\lambda + \mu) \left(\frac{\lambda x + \mu y}{\lambda + \mu} \right)^p \geq \frac{\lambda \mu (\lambda^{p-1} + \mu^{p-1})}{(\lambda + \mu)^p} |x - y|^p .$$

The proof then continues by constructing a more regular approximation of the extremal function, that is a certain step function adapted to a subtree of the mesure space and then a close study of the terms and their contribtion that appear in the first, combinatorial analytic, proof of the equality of the L^p Bellman function.

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